Bayesian estimation of Warner's randomized response technique with transmuted Kumaraswamy prior

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Abstract

In recent times, the Bayesian approach to randomized response technique has been used for estimating the population proportion, especially of respondents possessing sensitive attributes such as induced abortion, tax evasion, and shoplifting. This is done by combining suitable prior information about an unknown parameter of the population with the sample information for the estimation of the unknown parameter. In this study, the possibility of using a transmuted Kumaraswamy prior is raised, yielding a new Bayes estimator for estimating the population proportion of sensitive attributes for Warner's randomized response technique. Consequently, the proposed Bayes estimator with transmuted Kumaraswamy prior is compared with existing Bayes estimators developed with a simple beta and Kumaraswamy priors in terms of their mean square error. The proposed estimator competes well with the existing estimators for some values of population proportion π. The performances of Bayes estimators were also compared using some benchmark data.

Keywords: Bayesian estimation; mean square error; randomized response technique; sensitive attribute; transmuted Kumaraswamy prior

1. Introduction

In a direct questioning approach regarding sensitive attributes, one gets at times untruthful responses. Estimates of response categories are usually biased. So many reasons may be responsible for this, which include the legal status of sensitive characteristics a question may carry. To circumvent this problem, Warner (1965) proposed a randomized response technique (RRT) to gather information regarding sensitive attributes by ensuring the confidentiality of the respondents. The RRT was designed to increase the frequency of truthful answers, and increase the cooperation of respondents on sensitive questions, among others. Chaudhuri & Mukerjee (1988) provided insightful discussions on the RRT. Up till now, Warner's randomized response technique has been extended by many researchers. These include unrelated RRT (Horvitz *et al.*, 1967; Greenberg *et al.*, 1969), two alternative questions RRT (Folsom *et al.*, 1973), improved versions of RRT (Mangat & Singh, 1990; Mangat, 1994), a generalized RRT (Christofides, 2003), a stratified RRT (Kim & Warde, 2004), a tripartite RRT (Adepetun & Adebola, 2011), RRT for rare sensitive attributes (Singh & Tarray, 2014; Singh *et al.*, 2020), partial RRM for a rare sensitive attribute using Poisson distribution (Narjis & Shabbir, 2021b) among others.

In estimating the parameters of the randomized response model, one may use the classical (frequentist) approach or the Bayesian approach. The classical approach makes use of methods of moments and maximum likelihood. However, the classical approach may yield an unsatisfactory estimator (Hussain *et al*, 2014). At times, prior information about the unknown parameter may be available and can be incorporated with the sample information to estimate the unknown parameter. This is referred to as the Bayesian technique of estimation. Bayesian approach is employed to improve the efficiency of the estimator of unknown population proportion of sensitive attribute from the respondent due to the availability of prior information (Winkler & Franklin, 1979).

Kumaraswamy distribution is a special form of beta distribution that is widely applicable to a number of hydrological problems and many natural phenomena whose process values are bounded on both sides. It is more flexible than the beta distribution (Kumaraswamy, 1980) in many applications. However, the transmuted Kumaraswamy distribution, on the other hand, is a generalization of the Kumaraswamy distribution which offers even more flexibility in statistical modeling and applications than the Kumaraswamy distribution whenever the value of the transmuting parameter in the distribution is varied considerably between a closed interval [-1,1] (Khan *et al.*, 2016). In this paper, the possibility of using transmuted Kumaraswamy prior is raised. The primary focus of this paper is to propose Bayesian estimation of Warner's randomized response technique of a population proportion of respondents possessing a sensitive attribute using transmuted Kumaraswamy prior. This study complements the work of Winkler & Franklin (1979), Spurrier & Padgett (1980), Pitz (1980), Migon, H. & Tachibana (1997), Bar-Lev *et al.* (2003), Kim *et al.* (2006), Barabesi & Marcheselli (2006), Hussain & Shabbir (2009), Hussain *et al.* (2011), Hussain & Shabbir (2012), Hussain *et al* (2014), (Adepetun & Adewara, 2017, 2018), Narjis & Shabbir (2021a), Ahmed *et al.* (2021) in the Bayesian frameworks. The rest of the paper is arranged as follows. In Section 2, both the existing Bayesian estimation of Warner's randomized response technique and the proposed Bayesian estimator are presented. In section 3, we present an efficiency comparison of the proposed estimator with the existing ones at a fixed set of parameters in the prior distributions. Section 4 is the empirical applications while section 5 contains concluding remarks.

2. Bayesian estimation of Warner's randomized response technique

Prior information is of particular interest in the Bayesian randomized response sampling technique (Winkler & Franklin, 1979). Hussain & Shabbir (2009) argued that the Bayes estimator is more efficient than the maximum likelihood estimator due to the availability of prior information.

2.1 Warner's Randomized Response Technique

Suppose X is a count of respondents who possess the sensitive attribute in a random sample of size n. Then, X is distributed as binomial with parameters n and φ. That is, X ∼ Bin(n, φ), where φ is the probability of a yes response defined as

$$\phi = p\pi + (1-p)(1-\pi), \tag{1}$$

where p is the predetermined probability that the randomized device points to sensitive question (Warner, 1965). The parameter π is the true population proportion of respondents who possess the sensitive attribute.

The maximum likelihood estimator of π, denoted by $\hat{\pi}$, is

$$\hat{\pi} = \frac{\hat{\phi} - (1-p)}{2p-1}; \quad \text{provided } p \neq \frac{1}{2}.$$

Warner (1965) has shown that $\hat{\pi}$ is unbiased. The variance of $\hat{\pi}$ is

$$V(\hat{\pi}) = \frac{\pi(1-\pi)}{n} + \frac{p(1-p)}{n(2p-1)^2} \text{provided } p \neq \frac{1}{2} \tag{2}$$

The mean square error of $\hat{\pi}$ is equal to $V(\hat{\pi})$ in (2).

2.2 The Existing Bayesian Estimators

Warner's randomized response technique with a beta prior distribution for the estimation of population proportion of sensitive attribute yields posterior distribution which is conjugate to the prior distribution. Winkler & Franklin (1979) proposed Bayes estimator using a simple beta prior defined as

$$f_s(\pi,\ a,\ b) = \frac{1}{\beta(a,b)} \pi^{a-1}(1-\pi)^{b-1},$$

$$0 < \pi < 1 \text{ and } a, b > 0,$$

where the normalization constant $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\Gamma(a) = (a-1)!$. The authors derived a Bayes estimator using the probability of yes response in Warner's randomized response technique given in (1) as likelihood function. The Winkler and Franklin's Bayes estimator of π, denoted by $\hat{\pi}_{SW}$, is:

$$\hat{\pi}_{SW} = \frac{\sum_{i=0}^{x}\sum_{j=0}^{n-x}\left(\binom{x}{i}\binom{n-x}{j} f^{n-i-j}\beta(a+i+1, b+j)\right)}{\sum_{i=0}^{x}\sum_{j=0}^{n-x}\left(\binom{x}{i}\binom{n-x}{j} f^{n-i-j}\beta(a+i, b+j)\right)} \tag{3}$$

where $f = \frac{1-p}{2p-1}$ for $p \neq 0.5$. The mean square error of $\hat{\pi}_{SW}$ is

$$\text{MSE}(\hat{\pi}_{SW}) = E(\hat{\pi}_{SW} - \pi)^2 = \sum_{x=0}^{n} (\hat{\pi}_{SW} - \pi)^2 \phi^x (1-\phi)^{n-x} \tag{4}$$

Adepetun & Adewara (2017) proposed Bayes estimator using distribution proposed by Kumaraswamy (1980) as their alternative prior which was defined as:

$$f_k(\pi,\ b,\ c) = bc\pi^{b-1}(1-\pi^b)^{c-1}$$

$$b, c > 0\ ;\ \ 0 < \pi < 1.$$

With Kumaraswamy prior, a Bayes estimator ($\hat{\pi}_{KW}$) of π and its corresponding mean square error are defined as:

$$\hat{\pi}_{KW} = \frac{\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\left((-1)^k\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k} f^{n-i-j}\beta(b+bk+i+1, j+1)\right)}{\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\left((-1)^k\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k} f^{n-i-j}\beta(b+bk+i, j+1)\right)} \tag{5}$$

and

$$\text{MSE}(\hat{\pi}_{KW}) = E(\hat{\pi}_{KW} - \pi)^2 = \sum_{x=0}^{n} (\hat{\pi}_{KW} - \pi)^2 \phi^x (1-\phi)^{n-x} \tag{6}$$

respectively, where x is the number of respondents who have committed the sensitive attribute in a sample of size n.

2.3 The Proposed Bayesian Estimator

Khan *et al.* (2016) proposed a new three-parameter distribution which is a generalized two- parameter Kumaraswamy distribution, called the transmuted Kumaraswamy distribution. The probability density function of transmuted Kumaraswamy distribution is defined as

$$f_{TKw}(\pi,\ b,\ c,\ \lambda) = bc\pi^{b-1}\left(1-\pi^b\right)^{c-1}\left\{1-\lambda+2\lambda\left(1-\pi^b\right)^c\right\}; \tag{7}$$

$$b, c > 0;\ -1 \leq \lambda \leq 1$$

where λ is the transmuting parameter. The generalization was obtained by transforming the two parameter Kumaraswamy distribution through the quadratic rank transmuted map technique. The proposed transmuted Kumaraswamy distribution led to a better fit than the Kumaraswamy distribution. By combining the transmuted Kumaraswamy prior and the likelihood function of X, the joint distribution of X and π is obtained as

$$f_{TKw}(x,\pi) = f_{TKw}(\pi,\ b,\ c,\ \lambda)\, f(x \mid \pi). \tag{8}$$

The probability mass function of $X \mid \pi$ is

$$f(x \mid \pi) = \binom{n}{x} \phi^x (1-\phi)^{n-x}$$
$$= \binom{n}{x} (2p-1)^n \sum_{i=0}^{x} \sum_{j=0}^{n-x} \left(\binom{x}{i} \binom{n-x}{j} \times f^{n-i-j} \pi^i (1-\pi)^j \right) \quad (9)$$

for $x = 0, 1, 2, \ldots, n$ (Warner, 1965). The expression in (8) simplifies to

$$f_{TKw}(x,\pi) = bc\pi^{b-1} \left(1-\pi^b\right)^{c-1} \left\{1-\lambda+2\lambda\left(1-\pi^b\right)^c\right\} \binom{n}{x} (2p-1)^n$$
$$\times \sum_{i=0}^{x} \sum_{j=0}^{n-x} \left(\binom{x}{i} \binom{n-x}{j} f^{n-i-j} \pi^i (1-\pi)^j \right) \quad (10)$$

Using the expansion $\left(1-\pi^b\right)^{c-1} = \sum_{k=0}^{c-1} (-1)^k \binom{c-1}{k} \left(\pi^b\right)^k$, The joint probability density function can be obtained as

$$f_{TKw}(x,\pi) = \left(\text{bc} \binom{n}{x} (2p-1)^n - \lambda bc \binom{n}{x} (2p-1)^n + 2\lambda bc \binom{n}{x} \times (2p-1)^n \left(1-\pi^b\right)^c \right)$$
$$\times \sum_{i=0}^{x} \sum_{j=0}^{n-x} \sum_{k=0}^{c-1} (-1)^k \binom{x}{i} \binom{n-x}{j} \times \binom{c-1}{k} f^{n-i-j} \pi^{b+bk+i-1} (1-\pi)^j . \quad (11)$$

The proposed Bayes estimator is given as

$$\hat{\pi}_{TKw} = \int_0^1 \pi f_{TKw}(\pi|x)\, d\pi$$
$$= \frac{\begin{array}{c} \sum_{i=0}^{x} \sum_{j=0}^{n-x} \sum_{k=0}^{c-1} \left((-1)^k \binom{x}{i} \binom{n-x}{j} \binom{c-1}{k} f^{n-i-j} \beta(b+bk+i+1, j+1) \right) \\ -\lambda \sum_{i=0}^{x} \sum_{j=0}^{n-x} \sum_{k=0}^{c-1} \left((-1)^k \binom{x}{i} \binom{n-x}{j} \binom{c-1}{k} f^{n-i-j} \beta(b+bk+i+1, j+1) \right) \\ +2\lambda \sum_{i=0}^{x} \sum_{j=0}^{n-x} \sum_{k=0}^{c-1} \sum_{l=0}^{c} \left((-1)^{k+l} \binom{x}{i} \binom{n-x}{j} \binom{c-1}{k} \binom{c}{l} f^{n-i-j} \beta(b+bk+bl+i+1, j+1) \right) \end{array}}{\begin{array}{c} \sum_{i=0}^{x} \sum_{j=0}^{n-x} \sum_{k=0}^{c-1} \left((-1)^k \binom{x}{i} \binom{n-x}{j} \binom{c-1}{k} f^{n-i-j} \beta(b+bk+i, j+1) \right) \\ -\lambda \sum_{i=0}^{x} \sum_{j=0}^{n-x} \sum_{k=0}^{c-1} \left((-1)^k \binom{x}{i} \binom{n-x}{j} \binom{c-1}{k} f^{n-i-j} \beta(b+bk+i, j+1) \right) \\ +2\lambda \sum_{i=0}^{x} \sum_{j=0}^{n-x} \sum_{k=0}^{c-1} \sum_{l=0}^{c} \left((-1)^{k+l} \binom{x}{i} \binom{n-x}{j} \binom{c-1}{k} \binom{c}{l} f^{n-i-j} \beta(b+bk+bl+i, j+1) \right) \end{array}} \quad (12)$$

Consequently, the mean square error (MSE) of the proposed Bayes estimator $\hat{\pi}_{TKw}$ is

$$\text{MSE}(\hat{\pi}_{TKw}) = E(\hat{\pi}_{TKw} - \pi)^2$$
$$= \sum_{x=0}^{n} (\hat{\pi}_{TKw} - \pi)^2 \phi^x (1-\phi)^{n-x} \quad (13)$$

The detailed mathematical derivations of the proposed Bayes estimator are provided in the appendix.

It is noteworthy that the proposed transmuted Kumaraswamy Bayes estimator in (12) reduces to the existing Bayes estimator given in (5) when the transmuting parameter λ is zero.

3. Efficiency of the Proposed Estimator

In this section, the efficiency of the proposed Bayes estimator of π is presented based on the comparison of mean square error (MSE) of the proposed Bayes estimator and existing estimators of π through a simulation study. That is, MSE of $\hat{\pi}_{TKw}$, MSE of $\hat{\pi}_{Kw}$ and MSE of $\hat{\pi}_{Sw}$ will be compared for varying values of π.

3.1 Simulation Study

It is important to note that the proposed estimator is derived from a three-parameter transmuted

Table 1. Comparison of MSE of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ at varying values of π when $n = 20, x = 9, p = 0.2, p = 0.9$ and the true prior distribution is $Beta(2, 5)$

π	p = 0.2			p = 0.9		
	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)
0.1	7.11E-04	1.43E-03	1.35E-03	5.56E-19	7.54E-16	7.08E-16
0.15	2.56E-04	5.48E-04	5.12E-04	5.98E-17	3.65E-14	3.41E-14
0.2	8.71E-05	2.00E-04	1.86E-04	1.25E-14	8.92E-13	8.30E-13
0.25	2.78E-05	6.95E-05	6.42E-05	5.81E-13	1.34E-11	1.24E-11
0.3	8.22E-06	2.28E-05	2.09E-05	1.37E-11	1.38E-10	1.27E-10
0.35	2.22E-06	7.04E-06	6.41E-06	2.07E-10	1.06E-09	9.66E-10
0.4	5.37E-07	2.02E-06	1.82E-06	2.27E-09	6.42E-09	5.78E-09
0.45	1.12E-07	5.37E-07	4.78E-07	1.95E-08	3.16E-08	2.81E-08
0.5	1.91E-08	1.30E-07	1.14E-07	1.37E-07	1.30E-07	1.14E-07
0.55	2.32E-09	2.82E-08	2.41E-08	8.18E-07	4.52E-07	3.87E-07
0.6	1.28E-10	5.35E-09	4.45E-09	4.26E-06	1.34E-06	1.12E-06
0.65	1.28E-12	8.64E-10	6.86E-10	1.97E-05	3.38E-06	2.68E-06
0.7	1.35E-11	1.13E-10	8.31E-11	8.25E-05	7.03E-06	5.18E-06
0.75	8.95E-12	1.08E-11	6.89E-12	3.16E-04	1.13E-05	7.23E-06
0.8	3.18E-12	6.04E-13	2.59E-13	1.12E-03	1.16E-05	4.95E-06
0.85	7.69E-13	6.42E-15	4.12E-16	3.69E-03	2.52E-06	1.62E-07
0.9	1.33E-13	1.91E-15	5.99E-15	1.15E-02	1.81E-05	5.68E-05
0.95	1.63E-14	1.12E-15	1.89E-15	3.37E-02	3.21E-04	5.38E-04

Kumaraswamy distributed prior unlike the two existing estimators considered which were obtained from two-parameter beta and Kumaraswamy distributed priors.

Since some of the parameters of these estimators are from the prior distributions, setting the same values for the parameters of the three estimators might result in under or overestimation. This is bound to affect the efficiencies and give a false impression that an estimator is better than others. To avoid this situation, a simulation study is conducted for three cases to compare the efficiency of the proposed estimator with the other two existing Bayes estimators.

In the first case, samples are generated from a beta distribution with a fixed parameter set. The samples are then fitted separately with both Kumaraswamy and transmuted Kumaraswamy distributions. This allows for the determination of the parameter sets for the two prior distributions which might be considered to be equivalent to that of the fixed beta prior distribution. All other parameters are then set to specific values.

In the second and third cases, the process is repeated by generating samples from the Kumaraswamy distribution and the transmuted Kumaraswamy distributions respectively. Sample sizes are set to 100 and the process is replicated 2000 times for each case. All computations are executed using R development software. The distributions are fitted using the *AdequacyModel* package of the software. Figure 1 shows the plots of the fitted prior distributions and the simulated data from specific distributions which were used in obtaining the equivalent parameter sets for other distributions.

Table 2. Comparison of MSE of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ at varying values of π when $n = 100, x = 42, p = 0.2, p = 0.9$ and the true prior distribution is $Beta(2, 5)$

π	p = 0.2			p = 0.9		
	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)
0.1	1.42E-15	4.96E-14	4.65E-14	1.45E-77	1.99E-75	1.87E-75
0.15	4.33E-17	6.91E-16	6.46E-16	2.34E-68	9.03E-67	8.44E-67
0.2	9.42E-19	7.98E-18	7.42E-18	8.63E-61	1.41E-59	1.31E-59
0.25	1.53E-20	7.50E-20	6.93E-20	2.40E-54	1.98E-53	1.83E-53
0.3	1.89E-22	5.62E-22	5.16E-22	9.92E-49	4.55E-48	4.18E-48
0.35	1.76E-24	3.28E-24	2.98E-24	9.48E-44	2.56E-43	2.33E-43
0.4	1.22E-26	1.45E-26	1.31E-26	2.83E-39	4.65E-39	4.19E-39
0.45	6.17E-29	4.70E-29	4.18E-29	3.26E-35	3.31E-35	2.95E-35
0.5	2.22E-31	1.07E-31	9.40E-32	1.71E-31	1.07E-31	9.40E-32
0.55	5.45E-34	1.65E-34	1.41E-34	4.61E-28	1.77E-28	1.51E-28
0.6	8.79E-37	1.60E-37	1.33E-37	7.05E-25	1.59E-25	1.32E-25
0.65	8.78E-40	9.11E-41	7.23E-41	6.57E-22	8.32E-23	6.61E-23
0.7	5.07E-43	2.72E-44	2.01E-44	3.99E-19	2.57E-20	1.90E-20
0.75	1.55E-46	3.62E-48	2.31E-48	1.66E-16	4.58E-18	2.93E-18
0.8	2.25E-50	1.56E-52	6.68E-53	4.91E-14	3.99E-16	1.71E-16
0.85	1.35E-54	6.30E-58	4.04E-59	1.08E-11	5.88E-15	3.77E-16
0.9	2.72E-59	3.01E-62	9.45E-62	1.80E-09	2.31E-12	7.23E-12
0.95	1.43E-64	9.76E-67	1.64E-66	2.36E-07	1.85E-09	3.10E-09

In the first case where data were simulated from $Beta(2, 5)$, the Kumaraswamy distribution and transmuted Kumaraswamy which can be considered to appropriately fit the data set are $Kum(1.74, 6.64)$ and $TKum(1.63, 5.57, 0.02)$ respectively as shown in Figure 1(a). In the second case, appropriate fits were obtained by $Beta(2.56, 4.42)$ and $TKum(1.89, 4.70, -0.09)$ using data from $Kum(2, 5)$ while $Beta(2.38, 5.11)$ and $Kum(1.92, 6.47)$ appropriately fit data from $TKum(2, 5, 0.5)$ in the third case as shown in Figures 1(b) and (c) respectively.

3.2 Efficiency with Fixed Set of Parameters for Beta Prior

In ascertaining the performance of an estimator of a parameter, one may judge its performance based on its mean square error (MSE) compared to the MSEs of other estimators of the same parameter. The performance of the proposed estimator $\hat{\pi}_{TKw}$ of π is compared with the performance of estimators $\hat{\pi}_{SW}$ and $\hat{\pi}_{KW}$ based on their MSEs. Suppose the true prior distribution is $Beta(2, 5)$, the equivalent Kumaraswamy distribution is $Kum(1.74, 6.64)$ and the equivalent transmuted Kumaraswamy distribution is $TKum(1.63, 5.57, 0.02)$ as described in section (3.1). Using these prior distributions for correspond-

Table 3. Comparison of MSE of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ at varying values of π when $n = 20, x = 0.9, p = 0.2, p = 0.9$ and the true prior distribution is $Kum(2, 5)$

π	p = 0.2			p = 0.9		
	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)
0.1	8.36E-04	1.05E-03	1.32E-03	3.93E-18	5.51E-16	6.94E-16
0.15	3.06E-04	3.91E-04	5.01E-04	2.17E-18	2.60E-14	3.34E-14
0.2	1.06E-04	1.39E-04	1.82E-04	3.94E-15	6.20E-13	8.10E-13
0.25	3.46E-05	4.67E-05	6.26E-05	3.11E-13	8.98E-12	1.20E-11
0.3	1.06E-05	1.47E-05	2.04E-05	8.89E-12	8.92E-11	1.23E-10
0.35	2.97E-06	4.34E-06	6.21E-06	1.49E-10	6.55E-10	9.37E-10
0.4	7.59E-07	1.18E-06	1.76E-06	1.74E-09	3.73E-09	5.59E-09
0.45	1.72E-07	2.89E-07	4.60E-07	1.56E-08	1.70E-08	2.70E-08
0.5	3.34E-08	6.33E-08	1.09E-07	1.13E-07	6.33E-08	1.09E-07
0.55	5.21E-09	1.19E-08	2.29E-08	6.92E-07	1.92E-07	3.68E-07
0.6	5.63E-10	1.84E-09	4.18E-09	3.67E-06	4.61E-07	1.05E-06
0.65	2.50E-11	2.08E-10	6.34E-10	1.72E-05	8.15E-07	2.48E-06
0.7	6.43E-13	1.31E-11	7.46E-11	7.29E-05	8.15E-07	4.66E-06
0.75	3.00E-12	4.37E-14	5.84E-12	2.82E-04	4.58E-08	6.12E-06
0.8	1.61E-12	2.28E-13	1.79E-13	1.01E-03	4.36E-06	3.43E-06
0.85	4.69E-13	1.51E-13	2.72E-15	3.35E-03	5.94E-05	1.07E-06
0.9	9.02E-14	4.04E-14	7.76E-15	1.05E-02	3.83E-04	7.36E-05
0.95	1.19E-14	6.35E-15	2.17E-15	3.09E-02	1.81E-03	6.19E-04

ing estimators, MSE of each estimator is computed for various values of π and some p.

Table 1 presents MSEs of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ for small sample size ($n = 20$) at some values of p and π. It is observed at $p = 0.2$ that the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.70$ while MSE of $\hat{\pi}_{TKw}$ is less than MSE of $\hat{\pi}_{KW}$. However, Kumaraswamy beta estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.90 \leq \pi \leq 1$ while the proposed transmuted Kumaraswamy estimator $\hat{\pi}_{TKw}$ achieves the least MSE when $0.75 \leq \pi \leq 0.85$. At $p = 0.9$, the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.45$ while MSE of $\hat{\pi}_{TKw}$ is less than MSE of $\hat{\pi}_{KW}$. The Kumaraswamy beta estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.90 \leq \pi \leq 1$ while the proposed transmuted Kumaraswamy estimator $\hat{\pi}_{TKw}$ achieves the least MSE when $0.50 \leq \pi \leq 0.85$. The proposed estimator $\hat{\pi}_{TKw}$ performs better than $\hat{\pi}_{KW}$ and $\hat{\pi}_{SW}$ in terms of MSE when the value of π is moderately large.

Table 2 presents the MSEs of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ & $\hat{\pi}_{TKw}$ for large sample size ($n = 100$) at some values of p and π. It is observed at $p = 0.2$ that the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.40$ while MSE of $\hat{\pi}_{TKw}$ is less than MSE of $\hat{\pi}_{KW}$. The proposed transmuted Kumaraswamy estimator $\hat{\pi}_{TKw}$ achieves the least MSE when $0.45 \leq \pi \leq 0.85$ while Kumaraswamy beta estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.90 \leq \pi \leq 1$. The results are similar for $p = 0.9$. These results confirm that the proposed estimator $\hat{\pi}_{TKw}$ performs better than $\hat{\pi}_{KW}$ and $\hat{\pi}_{SW}$ in terms of MSE when the value of π is moderately large.

3.3 Efficiency with Fixed Set of Parameters for Kumaraswamy Prior

Suppose the true prior distribution is $Kum(2, 5)$, the equivalent Beta distribution is $Beta(2.56, 4.42)$

and the equivalent transmuted Kumaraswamy distribution is $TKum(1.89, 4.70, -0.09)$ as described in section 3.1. Using these prior distributions for corresponding estimators, MSE of each estimator is computed for various values of π and some p.

Table (3) presents MSEs of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ for for small sample sample size ($n = 20$) at some values of p and π when the true prior distribution is $Kum(2, 5)$. It is observed at $p = 0.2$ that the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.70$. However, the estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.70 \leq \pi \leq 0.75$ while the estimator $\hat{\pi}_{TKw}$ achieves the least MSE when $0.80 \leq \pi \leq 1$. At $p = 0.9$, the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.45$. The estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.50 \leq \pi \leq 0.75$ while the $\hat{\pi}_{TKw}$ achieves the least MSE when $0.80 \leq \pi \leq 1$.

Table 4 presents the MSEs of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ for large sample sample size ($n = 100$) at some values of p and π when the true prior distribution is $Kum(2, 5)$. It is observed at $p = 0.2$ that the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.25$. The estimator $\hat{\pi}_{TKw}$ achieves the least MSE when $0.80 \leq \pi \leq 1$ while Kumaraswamy beta estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.30 \leq \pi \leq 0.75$. At $p = 0.9$, the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.40$. The estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.45 \leq \pi \leq 0.75$ while the estimator $\hat{\pi}_{TKw}$ achieves the least MSE when $0.80 \leq \pi \leq 1$.

Table 4. Comparison of MSE of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ at varying values of π when $n = 100, x = 42, p = 0.2, p = 0.9$ and the true prior distribution is $Kum(2, 5)$

π	p = 0.2			p = 0.9		
	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)
0.1	8.34E-15	3.62E-14	4.56E-14	1.05E-77	1.46E-75	1.83E-75
0.15	1.79E-16	4.93E-16	6.32E-16	1.96E-68	6.45E-67	8.26E-67
0.2	3.08E-18	5.54E-18	7.24E-18	7.64E-61	9.77E-60	1.28E-59
0.25	4.24E-20	5.04E-20	6.75E-20	2.19E-54	1.33E-53	1.78E-53
0.3	4.61E-22	3.63E-22	5.01E-22	9.21E-49	2.94E-48	4.06E-48
0.35	3.89E-24	2.02E-24	2.89E-24	8.91E-44	1.58E-43	2.26E-43
0.4	2.50E-26	8.42E-27	1.26E-26	2.68E-39	2.70E-39	4.05E-39
0.45	1.19E-28	2.53E-29	4.02E-29	3.11E-35	1.79E-35	2.84E-35
0.5	4.04E-31	5.23E-32	8.99E-32	1.64E-31	5.23E-32	8.99E-32
0.55	9.50E-34	6.98E-35	1.34E-34	4.44E-28	7.48E-29	1.44E-28
0.6	1.47E-36	5.50E-38	1.25E-37	6.81E-25	5.47E-26	1.24E-25
0.65	1.42E-39	2.20E-41	6.68E-41	6.37E-22	2.01E-23	6.10E-23
0.7	7.97E-43	3.15E-45	1.80E-44	3.87E-19	2.98E-21	1.70E-20
0.75	2.38E-46	1.46E-50	1.96E-48	1.61E-16	1.86E-20	2.48E-18
0.8	3.38E-50	5.89E-53	4.63E-53	4.79E-14	1.51E-16	1.18E-16
0.85	1.97E-54	1.48E-56	2.67E-58	1.05E-11	1.38E-13	2.49E-15
0.9	3.91E-59	6.38E-61	1.22E-61	1.76E-09	4.88E-11	9.37E-12
0.95	2.02E-64	5.51E-66	1.88E-66	2.31E-07	1.04E-08	3.56E-09

3.4 Efficiency with Fixed Set of Parameters for Transmuted Kumaraswamy Prior

Suppose the true prior distribution is $TKum(2, 5, 0.5)$, the equivalent Beta distribution is $Beta(2.38, 5.11)$

and the equivalent Kumaraswamy distribution is $Kum(1.92, 6.47)$ as described in section 3.1. Using these prior distributions for corresponding estimators, MSE of each estimator is computed for various values of π and some p. Table 5 presents MSEs of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ for small sample size ($n = 20$) at some values of p and π when the true prior distribution is $TKum(2, 5, 0.5)$. It is observed at $p = 0.2$ that the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.65$. However, the estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.80 \leq \pi \leq 1.00$ while the estimator $\hat{\pi}_{TKw}$ achieves the least MSE when $0.70 \leq \pi \leq 0.75$. At $p = 0.9$, the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.40$. The estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.80 \leq \pi \leq 1.00$ while the $\hat{\pi}_{TKw}$ achieves the least MSE when $0.45 \leq \pi \leq 0.75$.

Table (6) presents the MSEs of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ for large sample size ($n = 100$) at some values of p and π when the true prior distribution is $TKum(2, 5, 0.5)$. It is observed at $p = 0.2$ that the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.25$. The estimator $\hat{\pi}_{TKw}$ achieves the least MSE when $0.30 \leq \pi \leq 0.75$ while Kumaraswamy beta estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.80 \leq \pi \leq 1$. At $p = 0.9$, the estimator $\hat{\pi}_{SW}$ achieves the least MSE when $0.10 \leq \pi \leq 0.35$. The estimator $\hat{\pi}_{KW}$ achieves the least MSE when $0.80 \leq \pi \leq 1$ while the estimator $\hat{\pi}_{TKw}$ achieves the least MSE when $0.40 \leq \pi \leq 0.75$.

Table 5. Comparison of MSE of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ at varying values of π when $n = 20, x = 0.9, p = 0.2, p = 0.9$ and the true prior distribution is $TKum(2, 5, 0.5)$

π	p = 0.2			p = 0.9		
	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)
0.1	7.25E-04	1.42E-03	9.41E-04	2.03E-18	7.48E-16	4.95E-16
0.15	2.62E-04	5.43E-04	3.48E-04	7.23E-18	3.62E-14	2.32E-14
0.2	8.92E-05	1.98E-04	1.22E-04	7.20E-15	8.84E-13	5.45E-13
0.25	2.85E-05	6.88E-05	4.06E-05	4.23E-13	1.32E-11	7.81E-12
0.3	8.47E-06	2.26E-05	1.26E-05	1.09E-11	1.37E-10	7.63E-11
0.35	2.30E-06	6.96E-06	3.64E-06	1.74E-10	1.05E-09	5.49E-10
0.4	5.61E-07	2.00E-06	9.61E-07	1.97E-09	6.34E-09	3.05E-09
0.45	1.19E-07	5.30E-07	2.28E-07	1.73E-08	3.12E-08	1.34E-08
0.5	2.06E-08	1.28E-07	4.75E-08	1.24E-07	1.28E-07	4.75E-08
0.55	2.59E-09	2.77E-08	8.29E-09	7.48E-07	4.44E-07	1.33E-07
0.6	1.62E-10	5.24E-09	1.12E-09	3.93E-06	1.31E-06	2.81E-07
0.65	1.80E-13	8.42E-10	9.64E-11	1.83E-05	3.29E-06	3.77E-07
0.7	1.11E-11	1.09E-10	2.11E-12	7.72E-05	6.79E-06	1.32E-07
0.75	8.10E-12	1.03E-11	5.49E-13	2.97E-04	1.08E-05	5.76E-07
0.8	2.97E-12	5.55E-13	7.49E-13	1.06E-03	1.06E-05	1.43E-05
0.85	7.31E-13	4.62E-15	2.85E-13	3.50E-03	1.81E-06	1.12E-04
0.9	1.28E-13	2.28E-15	6.23E-14	1.09E-02	2.16E-05	5.91E-04
0.95	1.58E-14	1.21E-15	8.83E-15	3.21E-02	3.44E-04	2.52E-03

3.5 Effect of Transmuting Parameter on the Efficiency of the Estimates with Transmuted Kumaraswamy Prior

It is pertinent to examine the effect of transmuting parameter on the estimator $\hat{\pi}_{TKw}$. Consider an

Table 6. Comparison of MSE of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ at varying values of π when $n = 100, x = 42, p = 0.2, p = 0.9$ and the true prior distribution is $TKum(2, 5, 0.5)$

π	p = 0.2			p = 0.9		
	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)	MSE($\hat{\pi}_{SW}$)	MSE($\hat{\pi}_{KW}$)	MSE($\hat{\pi}_{TKw}$)
0.1	5.18E-15	4.92E-14	3.25E-14	1.19E-77	1.98E-75	1.31E-75
0.15	1.19E-16	6.85E-16	4.39E-16	2.09E-68	8.96E-67	5.74E-67
0.2	2.17E-18	7.91E-18	4.88E-18	7.99E-61	1.39E-59	8.60E-60
0.25	3.11E-20	7.43E-20	4.38E-20	2.26E-54	1.96E-53	1.15E-53
0.3	3.49E-22	5.56E-22	3.11E-22	9.46E-49	4.51E-48	2.52E-48
0.35	3.02E-24	3.24E-24	1.69E-24	9.11E-44	2.53E-43	1.32E-43
0.4	1.98E-26	1.43E-26	6.88E-27	2.73E-39	4.59E-39	2.21E-39
0.45	9.59E-29	4.64E-29	2.00E-29	3.16E-35	3.27E-35	1.41E-35
0.5	3.32E-31	1.06E-31	3.93E-32	1.67E-31	1.06E-31	3.93E-32
0.55	7.91E-34	1.62E-34	4.86E-35	4.50E-28	1.73E-28	5.20E-29
0.6	1.24E-36	1.57E-37	3.36E-38	6.89E-25	1.56E-25	3.34E-26
0.65	1.21E-39	8.87E-41	1.02E-41	6.44E-22	8.10E-23	9.28E-24
0.7	6.85E-43	2.63E-44	5.09E-46	3.92E-19	2.49E-20	4.82E-22
0.75	2.06E-46	3.44E-48	1.84E-49	1.63E-16	4.36E-18	2.33E-19
0.8	2.95E-50	1.43E-52	1.93E-52	4.83E-14	3.67E-16	4.95E-16
0.85	1.73E-54	4.53E-58	2.80E-56	1.06E-11	4.22E-15	2.61E-13
0.9	3.46E-59	3.60E-62	9.83E-61	1.78E-09	2.76E-12	7.53E-11
0.95	1.80E-64	1.05E-66	7.66E-66	2.33E-07	1.98E-09	1.45E-08

experiment where the prior is assumed to be distributed as $TKum(2,\ 5,\ \lambda)$ and the transmuting parameter λ is varied on $\{-1,\ 1]$. Using these priors, different estimates are obtained using the estimators $\hat{\pi}_{TKw}$ and the corresponding MSEs are computed. The result obtained is shown in Figure (2). From the result, it is clear that the MSE reduces for all value of p when $\lambda \geq -0.5$. The greatest value of the MSE is obtained when $\lambda = -0.6$. The MSE increases as the λ increases from -0.8 to 0.6 but decreases with increase in λ every other interval.

4. Empirical applications

The study adopted benchmark data obtained by administering survey questionnaires on abortion and the use of contraceptives by some women in Akure metropolis, Nigeria (Adepetun and Adewara, 2017). A total of 300 questionnaires were administered with 279 returned. Generally, most single women in Nigeria avoid answering questions related to their sexual lives and such question as "have you committed abortion?" and "do you use a contraceptive during sex?". The questions are considered sensitive.

4.1 Abortion data

The data contains 275 valid responses with 113 of the respondents agreeing to have committed abortion. Having assumed priors of the Beta, Kumaraswamy and Transmuted Kumaraswamy distributed forms, the parameters of the distributions are obtained through bootstrapping. A random sample of size 50 is selected from the 275 observations and the proportion of respondents with the positive responses is determined. This process is repeated 2000 times. This is done to generate the best fit for the prior distributions.

Descriptive statistics of sample proportions obtained are presented in Table (7). The parameters

Table 7. Descriptive statistics of proportions of respondents with positive response to abortion questions

Mean	Median	Variance	Max	Min
0.4102	0.4000	0.00395	0.6400	0.2000

of the prior distributions are determined by fitting the prior distributions to the estimated sample proportions. The priors which best fit the data for the distributions considered are $Beta(24.594, 35.235)$, $Kum(5.595, 99.930)$ and $Tkum(7.283, 240.510, 0.800)$ respectively as shown in Figure (3). Using these fitted prior distributions, the MSE for the three estimators are obtained as shown in Table (8). It is observed from the table that mean square errors of the three estimators are very small. They are almost zero for each value of p. However, the estimator $\hat{\pi}_{SW}$ achieves the least MSE. The low performance of $\hat{\pi}_{TKw}$ can be attributed to the fact that estimated mean proportion is less than 0.50 as shown in Table (7)

Table 8. Comparison of MSE of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ when $n = 275$, $x = 113$ at varying values of p for the abortion data

p	MSE	MSE	MSE
0.7	1.41E-93	7.93E-93	8.10E-93
0.75	7.90E-96	3.85E-95	3.93E-95
0.8	3.81E-98	1.68E-97	1.72E-97
0.85	1.60E-100	6.60E-100	6.74E-100
0.9	5.91E-103	2.31E-102	2.36E-102

4.2 Contraceptive data

The data was obtained by asking the respondents the question “Do you use a contraceptive during sex?” and 259 valid responses with 125 admitting to not using contraceptives. A procedure similar to what was carried out in section 4.1 was also employed in obtaining the parameters of the priors for the contraceptive data. Descriptive statistics of sample proportions obtained are presented in Table 9.

Table 9. Descriptive statistics of proportions of respondents with positive response to use of contraceptive

Mean	Median	Variance	Max	Min
0.4848	0.4800	0.0039	0.7200	0.2800

Using these fitted prior distributions, shown in Figure 4, MSEs for the three estimators are obtained as shown in Table 10. It is observed from the table that all the three estimators perform well in terms of their MSEs. Their MSEs are infinitesimal, almost zero for each value of p. However, the estimator $\hat{\pi}_{SW}$ achieves the least MSE. The low performance of $\hat{\pi}_{TKw}$ can also be attributed to the fact that estimated mean proportion is less than 0.50 as shown in Table 9.

5. Conclusion

A Bayes estimator is proposed in this study for estimating population proportion of sensitive attribute with transmuted Kumaraswamy prior adopting Warner’s randomized response technique. Efficiency of

Table 10. Comparison of MSE of $\hat{\pi}_{SW}$, $\hat{\pi}_{KW}$ and $\hat{\pi}_{TKw}$ when $n = 259$, $x = 125$ at varying values of p for the Contraceptive data

p	MSE	MSE	MSE
0.7	1.79E-81	6.24E-81	7.50E-81
0.75	8.58E-82	2.50E-81	3.01E-81
0.8	3.87E-82	1.00E-81	1.20E-81
0.85	1.68E-82	3.98E-82	4.78E-82
0.9	7.15E-83	1.58E-82	1.90E-82

the proposed Bayesian estimator is considered with the existing Bayesian estimators developed with simple beta and Kumaraswamy priors through both simulation study and real life data on abortion and use of contraceptives among some women at varying values of fixed set of parameters in the priors and sample sizes. Findings from this study show that the proposed estimator is highly efficient at some high values of pi. At small values of π, the proposed estimator competes well with other estimators. Efficiency of the proposed estimator for fixed set of parameters in each prior was determined. Similarly, the effect of varying the values of the transmuting parameter of the newly introduced transmuted Kumaraswamy prior to mean square errors was also established as seen in Figure 2. Consequently, the proposed estimator is considered suitable for survey sampling of respondents with respect to sensitive attributes. Also, the mean square errors of the three competing Bayesian estimators are almost zero for the two benchmark real examples.

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Appendix

Derivations of the proposed Bayes estimator

Let $y = bc\binom{n}{x}(2p-1)^n$, the expression in (11) yields

$$f_{TKw}(x,\pi) = \left(y - \lambda y + 2\lambda y\left(1-\pi^b\right)^c\right)\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}(-1)^k\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\pi^{b+bk+i-1}(1-\pi)^j$$
$$= y\Bigg(\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}(-1)^k\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\pi^{b+bk+i-1}(1-\pi)^j$$
$$-\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}(-1)^k\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\pi^{b+bk+i-1}(1-\pi)^j$$
$$+2\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\sum_{l=0}^{c}(-1)^{k+l}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}\binom{c}{l}f^{n-i-j}\pi^{b+bk+bl+i-1}(1-\pi)^j\Bigg) \tag{14}$$

The marginal distribution of X is obtained from $f_{TKw}(x,\pi)$ by integrating it with respect to π as

$$f_{TKw}(x) = \int_0^1 f_{TKw}(x,\pi)\,d\pi$$
$$= y\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}(-1)^k\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\beta(b+bk+i,j+1)$$
$$-\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}(-1)^k\binom{x}{i}\binom{n-x}{j}\binom{c-1}{i}f^{n-i-j}\beta(b+bk+i,j+1)$$
$$+2\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\sum_{l=0}^{c}(-1)^{k+l}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}\binom{c}{l}f^{n-i-j}\beta(b+bk+bl+i,j+1) \tag{15}$$

where

$$\int_0^1 \pi^{b+bk+i-1}(1-\pi)^j\,d\pi = \beta(b+bk+i,j+1) = \frac{\Gamma(b+bk+i)\Gamma(j+1)}{\Gamma(b+bk+i+j+1)} \tag{16}$$

and

$$\int_0^1 \pi^{b+bk+bl+i-1}(1-\pi)^j\,d\pi = \beta(b+bk+bl+i,j+1) = \frac{\Gamma(b+bk+bl+i)\Gamma(j+1)}{\Gamma(b+bk+bl+i+j+1)} \tag{17}$$

The posterior distribution of π given X is derived as

$$f_{TKw}(\pi|x) = \frac{f_{TKw}(x,\pi)}{f_{TKw}(x)}$$
$$= \frac{\begin{array}{c}\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}(-1)^k U_1(n,x,b,c,i,j,k)\\ -\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}(-1)^k U_1(n,x,b,c,i,j,k)\\ +2\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\sum_{l=0}^{c}(-1)^{k+l} U_2(n,x,b,c,i,j,k,\ l)\end{array}}{\begin{array}{c}\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}(-1)^k T_1(n,x,b,c,i,j,k)\\ -\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}(-1)^k T_1(n,x,b,c,i,j,k)\\ +2\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\sum_{l=0}^{c}(-1)^{k+l} T_2(n,x,b,c,i,j,k,l)\end{array}} \tag{18}$$

where

$$U_1\left(n,x,b,c,i,j,k\right)=\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\pi^{b+bk+i-1}\left(1-\pi\right)^{j},$$

$$U_2\left(n,x,b,c,i,j,k,l\right)=\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}\binom{c}{l}f^{n-i-j}\pi^{b+bk+bl+i-1}\left(1-\pi\right)^{j},$$

$$T_1\left(n,x,b,c,i,j,k\right)=\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\beta\left(b+bk+i,j+1\right)$$

and

$$T_2\left(n,x,b,c,i,j,k,l\right)=\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}\binom{c}{l}f^{n-i-j}\beta\left(b+bk+bl+i,j+1\right).$$

The proposed Bayes estimator with transmuted Kumaraswamy prior is defined as

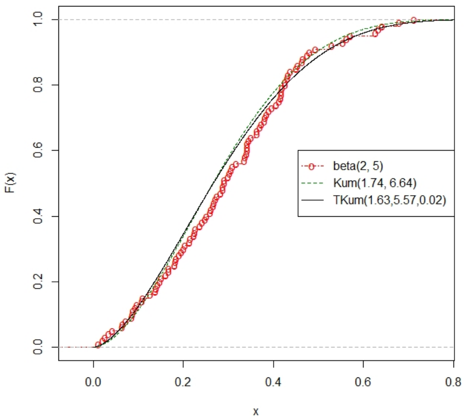
$$\hat{\pi}_{TKw}=\int_0^1\pi f_{TKw}\left(\pi|x\right)d\pi$$

$$=\frac{\begin{array}{c}\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\left((-1)^{k}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\int_0^1\pi^{b+bk+i}\left(1-\pi\right)^{j}d\pi\right.\\-\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\left((-1)^{k}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\int_0^1\pi^{b+bk+i}\left(1-\pi\right)^{j}d\pi\right)\\+2\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\sum_{l=0}^{c}\left((-1)^{k+l}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}\binom{c}{l}f^{n-i-j}\int_0^1\pi^{b+bk+i}\left(1-\pi\right)^{j}d\pi\right)\end{array}}{\begin{array}{c}\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\left((-1)^{k}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\int_0^1\pi^{b+bk+i}\left(1-\pi\right)^{j}d\pi\right)\\-\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\left((-1)^{k}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\int_0^1\pi^{b+bk+i}\left(1-\pi\right)^{j}d\pi\right)\\+2\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\sum_{l=0}^{c}\left((-1)^{k+l}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}\binom{c}{l}f^{n-i-j}\beta\int_0^1\pi^{b+bk+i}\left(1-\pi\right)^{j}d\pi\right)\end{array}} \quad (19)$$

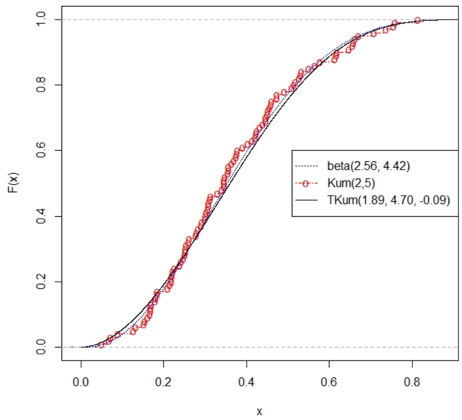
Evaluating the integral in (19) yields

$$\hat{\pi}_{TKw}=$$

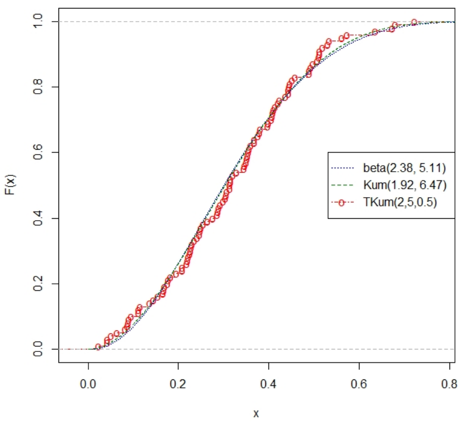
$$\frac{\begin{array}{c}\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\left((-1)^{k}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\beta\left(b+bk+i+1,j+1\right)\right)\\-\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\left((-1)^{k}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\beta\left(b+bk+i+1,j+1\right)\right)\\+2\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\sum_{l=0}^{c}\left((-1)^{k+l}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}\binom{c}{l}f^{n-i-j}\beta\left(b+bk+bl+i+1,j+1\right)\right)\end{array}}{\begin{array}{c}\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\left((-1)^{k}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\beta\left(b+bk+i,j+1\right)\right)\\-\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\left((-1)^{k}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}f^{n-i-j}\beta\left(b+bk+i,j+1\right)\right)\\+2\lambda\sum_{i=0}^{x}\sum_{j=0}^{n-x}\sum_{k=0}^{c-1}\sum_{l=0}^{c}\left((-1)^{k+l}\binom{x}{i}\binom{n-x}{j}\binom{c-1}{k}\binom{c}{l}f^{n-i-j}\beta\left(b+bk+bl+i,j+1\right)\right)\end{array}} \quad (20)$$



(a) Data Simulated from $beta(2, 5)$

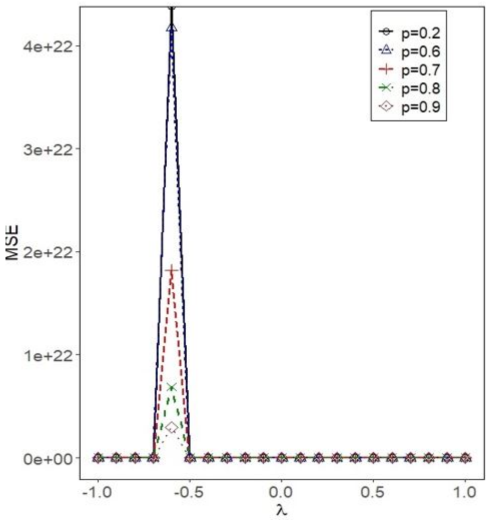


(b) Data Simulated from $Kum(2, 5)$

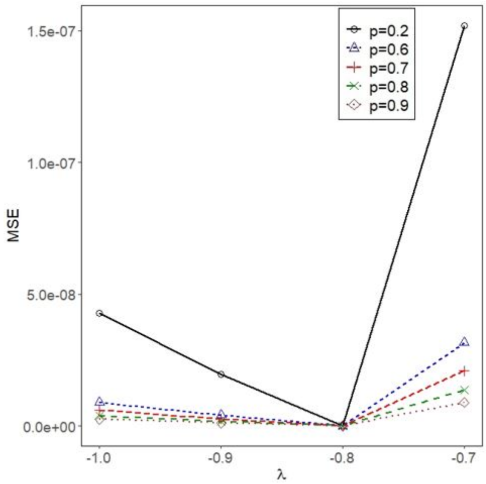


(c) Data Simulated from $TKum(2, 5, 0.5)$

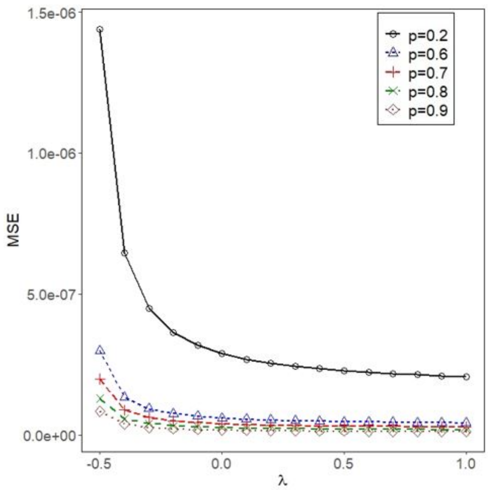
Fig. 1. Plots of fitted prior distributions and simulated data from (a) $beta(2, 5)$, (b) $Kum(2, 5)$ and (c) Transmuted $Kum(2, 5, 0.5)$.



(a) $\hat{\pi}_{TKw}$ for $\lambda \in [-1, 1]$



(b) $\hat{\pi}_{TKw}$ for $\lambda \in [-1, -0.7]$



(c) $\hat{\pi}_{TKw}$ for $\lambda \in [-0.5, 1]$

Fig. 2. Plot of MSE of $\hat{\pi}_{TKw}$ for various values of λ

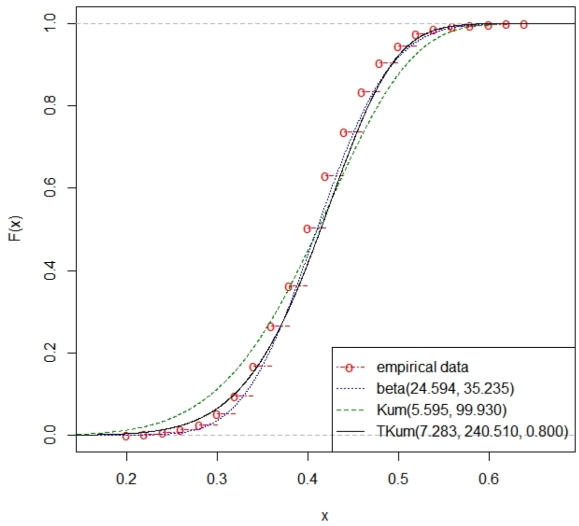


Fig. 3. Plot of the proportion of positive response obtained through bootstrapping and the fitted distributions for abortion data.

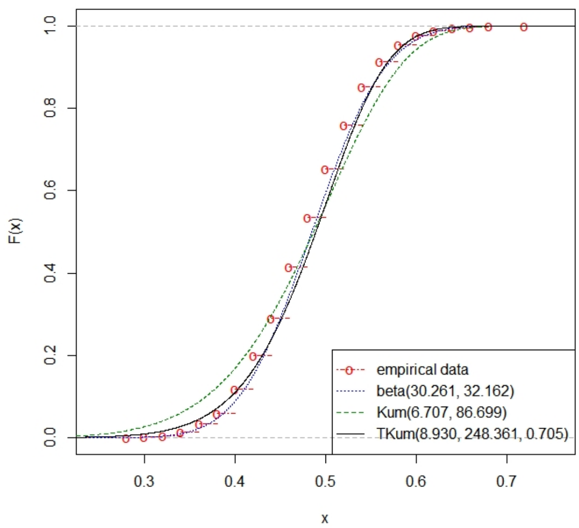


Fig. 4. Plot of the proportion of positive response obtained through bootstrapping and the fitted distributions for contraceptive data.