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Star complements in signed graphs with two symmetric eigenvalues

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Abstract

We consider signed graphs $\dot{G}$ whose spectra are comprised of exactly two (distinct) eigenvalues that differ only in sign, abbreviated to signed graphs with two symmetric eigenvalues. We obtain some relationships between such signed graphs and their star complements. Our results include structural examinations and constructions of infinite families of signed graphs with two symmetric eigenvalues. We also determine the bases for the eigenspaces of the eigenvalues of $\dot{G}$ in terms of the eigenspaces of its star complement. In particular, we consider the case in which a star complement has two symmetric eigenvalues, as well.

Keywords: Adjacency matrix; signed graph eigenvalue; signed graph eigenspace; signed graph decomposition; star complement

1. Introduction

A *signed graph* $\dot{G} = (G, \sigma)$ is an unsigned underlying graph $G = (V, E)$ with a *signature* σ that maps the set of edges into the multiplicative group $\{+1, -1\}$. The *order* of $\dot{G}$ refers to the cardinality of its vertex set. The edge set of $\dot{G}$ contains positive and negative edges. Evidently, every unsigned graph can be viewed as a signed graph without negative edges.

The *adjacency matrix* $A_{\dot{G}}$ is the adjacency matrix of G in which all ones related to negative edges are replaced with minus ones. The *eigenvalues* of $\dot{G}$ are the eigenvalues of A_G.

A characterization of signed graphs having a comparatively small number of eigenvalues is listed as an open problem in (Belardo *et al.*, 2018); of course, we refer to distinct eigenvalues. It is known that a signed graph with 2 eigenvalues must be regular; moreover, strongly regular in the sense of (Stanić II, 2019). Those with vertex degree at most 4 are determined in (Hou *et al.*, 2019; McKee and Smyth, 2007; Stanić, 2020). It is worth mentioning that signed graphs with 2 eigenvalues play a crucial role in Huang's resolution of the Sensitivity Conjecture (Huang, 2019); see also (Stanić I, 2019). Some particular constructions can be found in (Ramezani *et al.*, 2022; Stanić I, 2019; Stanić II, 2019).

In this study we focus our attention on signed graphs with exactly 2 eigenvalues $\pm\theta$, with $\theta \in \mathbb{R}$. For short, we abbreviate these signed graphs to *signed graphs with 2 symmetric eigenvalues*. We consider their decompositions into 2 induced subgraphs such that one of them is the star complement for one or both eigenvalues. We call such a decomposition a *star complementary decomposition*. Other kinds of graph decompositions can be found in (Romero-Valencia *et al.*, 2019). We establish some structural and spectral relations between a star complement and the other subgraph which can also be considered as the relations between the signed graph under consideration and its star complement. Then we use star complements to construct infinite families of signed graphs with 2 symmetric eigenvalues, and we also use the eigenspace bases of star complements to build the bases for the eigenspaces of the eigenvalues $\pm\theta$. Throughout the paper we frequently consider the particular case in which a star complement is also a signed graph with 2 symmetric eigenvalues. Our results can be encapsulated into those concerning signed graphs whose spectra are symmetric (with respect to the origin). We recall that in the framework of unsigned graph such graphs are fully characterised since a graph has a symmetric spectrum if and only if it is bipartite. In the domain of signed graphs the problem of characterizing such signed graphs is more intriguing. Some recent results can be found in (Anđelić *et al.*, 2020; Belardo *et al.*, 2018; Ghorbani *et al.*, 2020; Hou *et al.*, 2019; Ramezani *et al.*, 2022; Stanić I, 2019; Stanić II, 2019; Stanić, 2020).

Here is the contents of the remaining sections. The next section contains terminology and notation along with some known results. Section 3 is reserved for the structural and spectral examinations, and constructions of some examples. Eigenspace bases are established in Section 4.

2. Preliminaries

For the adjacent vertices i, j of a signed graph, we write $i \sim j$. The sign of the corresponding edge is emphasized by $i \overset{+}{\sim} j$ or $i \overset{-}{\sim} j$.

If there is a monomial $(0, 1, -1)$-matrix D such that $A_{\dot{H}} = D^{-1}A_{\dot{G}}D$ holds for some signed graphs $\dot{G}$ and $\dot{H}$, then we say that these signed graphs are *switching isomorphic*. Evidently, switching isomorphic signed graphs have the common spectrum. We say that signed graphs are *cospectral* if they are not switching isomorphic, but have the same spectrum. Moreover, in this study switching isomorphic signed graphs are mutually identified. The adjacency matrix $-A_{\dot{G}}$ determines the *negation* of $\dot{G}$, usually denoted by $-\dot{G}$. A (not necessarily induced) cycle contained in a signed graph is *positive* (resp. *negative*) if it has an even (odd) number of negative edges.

Let θ denote an arbitrary eigenvalue of $\dot{G}$, and let k denote its multiplicity. A subset X of $V(\dot{G})$ is a *star set* for θ if the size of X is k and θ does not occur in the spectrum of $\dot{G} - X$. Further, $\dot{G} - X$ is abbreviated to be a *star complement* for θ. The following generalization of the Reconstruction Theorem (Cvetković *et al.*, 2004, Theorem 5.1.7) can be found in (Ramezani *et al.*, 2020; Stanić I, 2019).

Theorem 2.1. *Let*

$$A_{\dot{G}} = \begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}, \tag{1}$$

where A_X is the $k \times k$ adjacency matrix of the subgraph determined by $X \subset V(\dot{G})$. Then X is a star set for θ if and only if θ does not appear in the spectrum of C and

$$\theta I - A_X = B^\top(\theta I - C)^{-1}B. \tag{2}$$

The eigenspace of θ consists of

$$\begin{pmatrix} \mathbf{x} \\ (\theta I - C)^{-1}B\mathbf{x} \end{pmatrix}, \tag{3}$$

for $\mathbf{x} \in \mathbb{R}^k$.

In relation to the previous theorem, we define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top(\theta I - C)^{-1}\mathbf{y}.$$

Also, for $u \in X$, $\mathbf{b}_u$ stands for the characteristic vector of the neigbourhood of u in the signed graph $\dot{G} - X$. Then Equation (2) yields:

$$\langle \mathbf{b}_u, \mathbf{b}_v \rangle = \begin{cases} \theta & \text{if } u = v \\ 0 & \text{if } u \not\sim v,\ u \neq v \\ 1 & \text{if } u \overset{-}{\sim} v \\ -1 & \text{if } u \overset{+}{\sim} v. \end{cases}$$

For a signed graph $\dot{H}$, an isolated vertex u and $U \subseteq V(\dot{H})$, we write $\dot{H}(U)$ for a signed graph gained by inserting an edge between u and every vertex of U. An edge between u and U is allowed to be of any sign. Next u, U and $\dot{H}(U)$ are said to be a *good vertex*, a *good set* and a *good extension* for θ whenever θ does not appear in the spectrum of $\dot{H}$, but does appear in that of $\dot{H}(U)$. It follows from Theorem 2.1 that u is good for θ if and only if $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \theta$. We say that the vertices u, v are *compatible* for θ if both are good and $\langle \mathbf{b}_u, \mathbf{b}_v \rangle \in \{0, 1, -1\}$. If so, then we also say that the corresponding sets (U for u and V

for v) are compatible. If u, v are good and $\dot{H}(U, V)$ is obtained by joining both u and v to $\dot{H}$ according to their characteristic vectors, then we deduce from Theorem 2.1 that $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = 0$ (resp. $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = 1$, $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = -1$) if and only if the multiplicity of θ in $\dot{H}(U, V)$ is two and also $u \nsim v$ (resp. $u \overset{-}{\sim} v$, $u \overset{+}{\sim} v$). In addition, from the same theorem we see that a set X whose elements are good, considered separately and in all possible pairs, induces a good extension, say $\dot{G}$, where X and $\dot{H}$ can be seen as a star set and the corresponding star complement for θ, respectively.

3. Star complementary decompositions of signed graphs with 2 symmetric eigenvalues

We begin with the following contribution to the matrix theory.

Theorem 3.1. *Let*

$$M = \begin{pmatrix} N_1 & B^\intercal \\ B & N_2 \end{pmatrix}$$

be a $2n \times 2n$ real symmetric matrix with exactly 2 eigenvalues θ and $-\theta$. If the $n \times n$ principal submatrix N_2 avoids $\pm\theta$ in the spectrum, then N_1 and $-N_2$ are similar.

Proof. Since the minimal polynomial of M is

$$x^2 - \theta^2 \tag{4}$$

we have $M^2 - \theta^2 I = O$. Concerning the top-right block of M^2, we get $N_1 B^\intercal = -B^\intercal N_2$. If $\mathbf{x}$ is an eigenvector of N_2 afforded by ν, then we have

$$N_1 B^\intercal \mathbf{x} = -B^\intercal N_2 \mathbf{x} = -\nu B^\intercal \mathbf{x}, \tag{5}$$

which means that $B^\intercal \mathbf{x}$ is an eigenvector of N_1 afforded by $-\nu$; note that $B^\intercal \mathbf{x} = \mathbf{0}$ leads to $\nu \in \{-\theta, \theta\}$ as

$$M \begin{pmatrix} \mathbf{0} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ N_2 \mathbf{x} \end{pmatrix} = \nu \begin{pmatrix} \mathbf{0} \\ \mathbf{x} \end{pmatrix}.$$

Since the vectors $B^\intercal \mathbf{x}$ belong to the image of $B^\intercal$, we get that $B^\intercal$ is invertible, which leads to $N_1 = -B^\intercal N_2 (B^\intercal)^{-1}$, and we are done. □

Consequently, if N_2 avoids $\pm\theta$ in the spectrum, N_1 also avoids these eigenvalues. We now proceed with signed graphs.

Corollary 3.2. *If $\dot{G}$ is a signed graph with eigenvalues $\pm\theta$ and a common star complement $\dot{H}$ for these eigenvalues, then the adjacency matrix of the signed graph $\dot{X}$ determined by the star set is similar to $A_{-\dot{H}}$ and corresponds to the star complement for $\pm\theta$.*

Proof. The fact that $A_{\dot{X}}$ is similar to $A_{-\dot{H}}$ follows from Theorem 3.1. Since θ and $-\theta$ do not belong to the spectrum $\dot{X}$, we get that $\dot{X}$ is a star complement for both eigenvalues. □

Corollary 3.3. *Under the notation of Corollary 3.2, the following statements hold:*

(i) If there is no star complement for $\pm\theta$ which is cospectral with $\dot{H}$, then the signed graph $\dot{X}$ is a negation of $\dot{H}$;

(ii) The vertices of $\dot{G}$ can be arranged in such a way that B is symmetric.

Proof. Under the assumption of this corollary, $A_{-\dot{H}}$ is a permutationally similar to $A_{\dot{H}}$, which leads to the assertion of (i).

Next, the vertices of $\dot{G}$ can be arranged such that

$$A_{\dot{G}} = \begin{pmatrix} -A_{\dot{H}} & B^\intercal \\ B & A_{\dot{H}} \end{pmatrix}.$$

Since $-\dot{H}$ is also a star complement for $\pm\theta$, we have that $\mathbf{b}_u$ determines the neighbourhood in $\dot{H}$ of a vertex $u \in V(-\dot{H})$ if and only if $\mathbf{b}_u^\intercal$ determines the neighbourhood in $-\dot{H}$ of the copy of u in $V(\dot{H})$. This implies $B = B^\intercal$. □

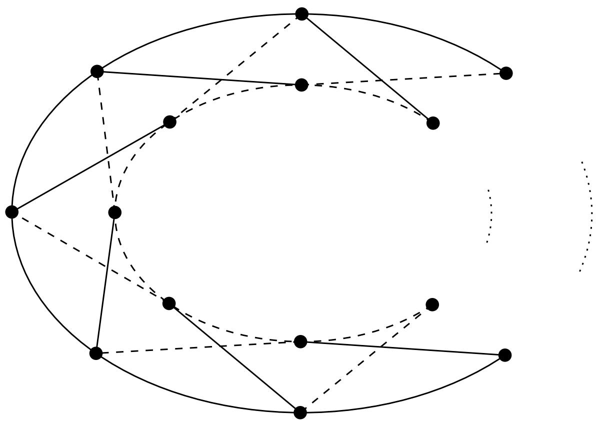

Fig. 1. The signed graph with spectrum $\left[2^t, (-2)^t\right]$, $t \geq 3$.

Example 3.4. We know from (Hou *et al.*, 2019; Stanić, 2020) that the signed graph of Fig. 1 has spectrum $\left[2^t, (-2)^t\right]$, where $t \geq 3$. According to the latter reference, this is the signed line graph of a signed graph derived from the unsigned cycle C_t by adding t negative edges in such a way that each of them is incident with a different pair of adjacent vertices of C_t.

By (Cvetković *et al.*, 2004, Theorem 5.1.6), a star complement can be taken to be connected. If so, then it can be seen that a connected star complement for both eigenvalues ± 2 is either a negative quadrangle with two paths of arbitrary lengths attached at its non-adjacent vertices, the graph obtained by attaching two hanging edges at a pendant vertex of a path or (for t even) a negative cycle $\dot{C}_t$. For t odd a positive (resp. negative) cycle $\dot{C}_t$ is a star complement for -2 (2). If $\dot{H}$ is one of star complements for both eigenvalues, then the star set induces the signed graph that switches to $\dot{H}$, as the negation of an even cycle switches to itself. If $\dot{H}$ is an odd cycle, then the star set induces the signed graph that switches to $-\dot{H}$.

If, in particular, $\dot{G}$ is a complete signed graph of order n with eigenvalues $\pm\theta$, then its adjacency matrix is the so-called symmetric conference matrix, i.e., it satisfies $A_{\dot{G}}^2 = A_{\dot{G}}A_{\dot{G}}^{\intercal} = (n-1)I$. In this case the (unsigned) graph induced by negative edges is called (by Seidel) a *strong graph*. (In general, the graph induced by negative edges is strong if $(A_{\dot{G}} - \lambda_1 I)(A_{\dot{G}} - \lambda_2 I) = (n-1+\lambda_1\lambda_2)J$, for $\lambda_1, \lambda_2 \in \mathbb{R}$, and our case is obtained for $\lambda_1 = -\lambda_2 = \theta = \sqrt{n-1}$.) An equivalence class of strong graphs obtained from $\dot{G}$ with 2 symmetric eigenvalues coincides with the regular two-graph equivalence class (Seidel, 1973). It is known that a graph is strong if it is strongly regular and satisfies $n = 2(r - \lambda)$; r is the vertex degree and λ is one of the remaining eigenvalues. Also, complete bipartite graphs and cones over conference graphs are strong. We know from (Haemers and Higman, 1989) that if $\dot{G}$ is complete with 2 symmetric eigenvalues and $\dot{H}$ is a common star complement for these eigenvalues, then $\dot{H}$ switches to a signed graph in which negative edges give rise to a conference graph, along with another unconfirmed possibility for $\dot{H}$. This brings us to the following example.

Example 3.5. Let $\dot{G}$ be a complete signed graph of order n that switches to a signed graph in which negative edges give rise to a strongly regular graph satisfying $n = 2(r - \lambda)$. Then $\dot{G}$ contains a star complement that switches to a signed graph in which negative edges give rise to a conference graph. The Petersen graph or its complement and the strongly regular graphs with 26 vertices serve as examples for strongly regular graphs that arise from $\dot{G}$ in the previous setting.

We proceed with some particular constructions.

Theorem 3.6. *For a signed graph $\dot{H}$ with eigenvalues $\pm\lambda$ and whose vertices are labelled by $1, 2, \ldots, t$, let $N(i)$ stand for the set of neighbours of a vertex i of $\dot{H}$. The vertex u_i and the corresponding set U_i are good for $\dot{H}$ being a star complement for θ whenever*

(i) $\theta = \sqrt{\lambda^2 + 1}$ and $U_i = \{i\}$, for $1 \leq i \leq t$ or

(ii) $\theta = \lambda\sqrt{2}$ *and* $U_i = N(i)$ *with* $\sigma(u_i j) = \sigma(ij)$ $(j \in N(i))$*, for* $1 \leq i \leq t$.

All vertices of (i) and (ii) are mutually compatible, with $u_i \nsim u_j$ *(resp.* $u_i \overset{+}{\sim} u_j$*,* $u_i \overset{-}{\sim} u_j$*) if* $i \nsim j$ *(*$i \overset{-}{\sim} j$*,* $i \overset{+}{\sim} j$*).*

Proof. Since the minimal polynomial of $A_{\dot{H}}$ is given by (4), we get (cf. (Cvetković *et al.*, 2004, Proposition 5.1.11))

$$(\theta I - A_{\dot{H}})^{-1} = \frac{1}{\theta^2 - \lambda^2}(A_{\dot{H}} + \theta I). \tag{6}$$

For (i) we have

$$\begin{aligned}\langle \mathbf{b}_{u_i}, \mathbf{b}_{u_i}\rangle &= \mathbf{b}_{u_i}^{\intercal}\frac{1}{\theta^2 - \lambda^2}(A_{\dot{H}} + \theta I)\mathbf{b}_{u_i} = \mathbf{b}_{u_i}^{\intercal}(A_{\dot{H}} + \sqrt{\lambda^2+1}I)\mathbf{b}_{u_i}\\ &= \mathbf{b}_{u_i}^{\intercal}A_{\dot{H}}\mathbf{b}_{u_i} + \sqrt{\lambda^2+1}\mathbf{b}_{u_i}^{\intercal}I\mathbf{b}_{u_i} = \sqrt{\lambda^2+1},\end{aligned}$$

so U_i is good.

Concerning the compatibility, we get

$$\begin{aligned}\langle \mathbf{b}_{u_i}, \mathbf{b}_{u_j}\rangle &= \mathbf{b}_{u_i}^{\intercal}\frac{1}{\theta^2 - \lambda^2}(A_{\dot{H}} + \theta I)\mathbf{b}_{u_j} = \mathbf{b}_{u_i}^{\intercal}(A_{\dot{H}} + \sqrt{\lambda^2+1}I)\mathbf{b}_{u_j}\\ &= \begin{cases} 0 \text{ if } i \nsim j\\ -1 \text{ if } i \overset{+}{\sim} j\\ 1 \text{ if } i \overset{-}{\sim} j,\end{cases}\end{aligned}$$

which completes the first item.

For (ii) we have

$$\begin{aligned}\langle \mathbf{b}_{u_i}, \mathbf{b}_{u_i}\rangle &= \mathbf{b}_{u_i}^{\intercal}\frac{1}{\theta^2 - \lambda^2}(A_{\dot{H}} + \theta I)\mathbf{b}_{u_i} = \mathbf{b}_{u_i}^{\intercal}(A_{\dot{H}} + \lambda\sqrt{2}I)\mathbf{b}_{u_i}\\ &= \frac{1}{\lambda^2}\mathbf{b}_{u_i}^{\intercal}A_{\dot{H}}\mathbf{b}_{u_i} + \frac{\sqrt{2}}{\lambda}\mathbf{b}_{u_i}^{\intercal}I\mathbf{b}_{u_i} = \lambda\sqrt{2},\end{aligned}$$

since

$$\mathbf{b}_{u_i}^{\intercal}A_{\dot{H}}\mathbf{b}_{u_i} = \big(\mathbf{b}_{u_i}\cdot\mathbf{b}_{u_1}, \mathbf{b}_{u_i}\cdot\mathbf{b}_{u_2}, \ldots, \mathbf{b}_{u_i}\cdot\mathbf{b}_{u_t}\big)\mathbf{b}_{u_i} = \big(0, 0, \ldots, 0, \mathbf{b}_{u_i}\cdot\mathbf{b}_{u_i}, 0, 0, \ldots, 0\big)\mathbf{b}_{u_i} = 0.$$

So U_i is good.

Concerning the compatibility, we get

$$\begin{aligned}\langle \mathbf{b}_{u_i}, \mathbf{b}_{u_j}\rangle &= \frac{1}{\lambda^2}\mathbf{b}_{u_i}^{\intercal}A_{\dot{H}}\mathbf{b}_{u_j} + \frac{\sqrt{2}}{\lambda}\mathbf{b}_{u_i}^{\intercal}I\mathbf{b}_{u_j}\\ &= \begin{cases} 0 \text{ if } i \nsim j\\ -1 \text{ if } i \overset{+}{\sim} j\\ 1 \text{ if } i \overset{-}{\sim} j,\end{cases}\end{aligned}$$

since $\mathbf{b}_{u_i}^{\intercal}A_{\dot{H}}\mathbf{b}_{u_j}$ is 0 (resp. $-\lambda^2$, λ^2) if $i \nsim j$ ($i \overset{+}{\sim} j$, $i \overset{-}{\sim} j$) and $\mathbf{b}_{u_i}^{\intercal}I\mathbf{b}_{u_j} = 0$ for $i \neq j$. □

As a consequence, we have the following.

Theorem 3.7. *A signed graph* $\dot{H}$ *with eigenvalues* $\pm\lambda$ *is a star complement in the signed graph* $\dot{G}$ *determined by*

$$A_{\dot{G}} = \begin{pmatrix} -A_{\dot{H}} & I\\ I & A_{\dot{H}}\end{pmatrix} \text{ or } A_{\dot{G}} = \begin{pmatrix} -A_{\dot{H}} & A_{\dot{H}}\\ A_{\dot{H}} & A_{\dot{H}}\end{pmatrix}$$

for $\pm\theta$ *where in the former case* $\theta = \sqrt{\lambda^2+1}$ *and in the latter case* $\theta = \lambda\sqrt{2}$.

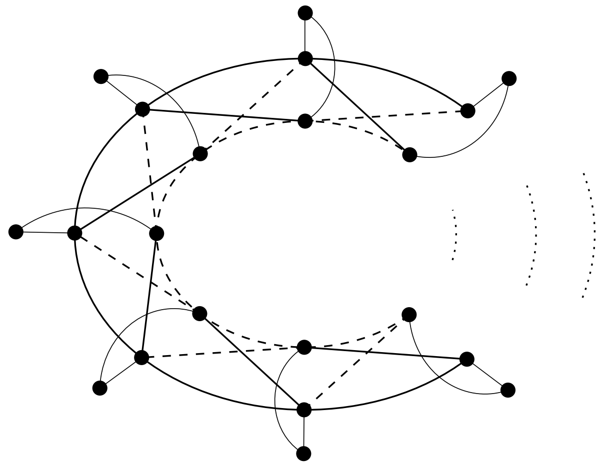

Fig. 2. The signed graph with spectrum $\left[\sqrt{6}^t, 0^t, -\sqrt{6}^t\right]$, $t \geq 3$.

Proof. The result follows by replacing θ with $-\theta$ in the proof of Theorem 3.6. $\square$

In (Ramezani *et al.*, 2022) constructions of non-regular signed graphs having just 3 eigenvalues are considered. Here we proceed with a construction of a family of such signed graphs which may be interesting because, apart from non-regularity, it has another relevant property: the eigenvalues are equal in multiplicity.

Theorem 3.8. *The signed graph $\dot{G}$ illustrated in Fig. 2 has spectrum* $\left[\sqrt{6}^t, 0^t, -\sqrt{6}^t\right]$, $t \geq 3$.

Proof. Let $\dot{H}$ denote the signed graph of Fig. 1. If A is its adjacency matrix, then $(\pm\sqrt{6}I - A)^{-1} = \frac{1}{2}(A \pm \sqrt{6}I)$. If u is a vertex adjacent to the pair of vertices of $\dot{H}$ with 4 common neighbours (as in the figure) then

$$\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mathbf{b}_u^\intercal \frac{1}{2}(A \pm \sqrt{6}I)\mathbf{b}_u = \frac{1}{2}\left(\mathbf{b}_u^\intercal A \mathbf{b}_u \pm \sqrt{6}\mathbf{b}_u^\intercal I \mathbf{b}_u\right) = \pm\sqrt{6},$$

as $\mathbf{b}_u^\intercal A \mathbf{b}_u = 0$, so u is a good vertex for $\dot{H}$ and $\pm\sqrt{6}$. Similarly, if u, v are good vertices, then $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = 0$, as $\mathbf{b}_u^\intercal A \mathbf{b}_v = \mathbf{b}_u^\intercal I \mathbf{b}_v = 0$. Therefore u, v are compatible as non-neighbours. It follows that the multiplicity of $\pm\sqrt{6}$ in $\dot{G}$ is t, so it remains to prove that zero appears with the same multiplicity. It is sufficient to show that zero shares the same star complement in $\dot{G}$. As before we get $(0I - A)^{-1} = \frac{1}{4}A$ (which can also be seen by considering the minimal polynomial of A). We further obtain $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \langle \mathbf{b}_u, \mathbf{b}_v \rangle = 0$, and the result follows. $\square$

4. Eigenspaces of signed graphs with 2 symmetric eigenvalues

Here we denote by $\dot{G}$ an arbitrary signed graph with eigenvalues $\pm\theta$ and assume that its adjacency matrix is given by Equation (1). Let further $\dot{H}$ stand for a star complement for $\pm\theta$. According to Equation (5), if $\mathbf{x}$ is an eigenvector of $\dot{H}$, then we have that $B^\intercal \mathbf{x}$ figures as an eigenvector of A_X. By (3), the eigenspace of $\pm\theta$ is spanned by the vectors $\begin{pmatrix} \mathbf{y}_i \\ (\pm\theta I - A_{\dot{H}})^{-1} B \mathbf{y}_i \end{pmatrix}$, $1 \leq i \leq t$, where $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_t$ make a basis of $\mathbb{R}^t$. One choice for the $\mathbf{y}_i$s is the canonical basis of $\mathbb{R}^t$ in which case we get

$$\begin{pmatrix} \mathbf{e}_i \\ (\pm\theta I - A_{\dot{H}})^{-1} \mathbf{b}_i^\intercal \end{pmatrix}, \; 1 \leq i \leq t, \tag{7}$$

where $\mathbf{b}_i^\intercal$ is the ith column of $B^\intercal$. Another choice is to take $\mathbf{y}_i = B^\intercal \mathbf{x}_i$ where $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_t$ make the full system of linearly independent eigenvectors of $\dot{H}$. In this case we get

$$\begin{pmatrix} B^\intercal \mathbf{x}_i \\ (\pm\theta I - A_{\dot{H}})^{-1} B B^\intercal \mathbf{x}_i \end{pmatrix}, \; 1 \leq i \leq t. \tag{8}$$

Indeed, $B^\intercal \mathbf{x}_i$ are linearly independent which can be seen by observing the proof of Theorem 3.1.

We record the previous discussion as the following statement.

Theorem 4.1. *Suppose that $\dot{G}$ is a connected signed graph whose adjacency matrix is given by Equation* (1) *and whose spectrum is $[\theta^t, -\theta^t]$. If there is a star complement for both eigenvalues of $\dot{G}$, then the eigenspaces of its eigenvalues are determined by any of bases* (7) *or* (8).

Observe that both bases are determined irrespective of A_X. By virtue of Corollary 3.3, there is a switching of $\dot{G}$ is which B is symmetric, and in this case the bases can be additionally simplified.

We proceed with the case in which $\dot{H}$ has 2 symmetric eigenvalues.

Corollary 4.2. *Suppose that $\dot{G}$ is a connected signed graph whose adjacency matrix is given by Equation* (1) *and whose spectrum is $[\theta^t, -\theta^t]$. If $\dot{G}$ contains a star complement (for $\pm\theta$) with spectrum $[\lambda, -\lambda]$, then the eigenspaces of the eigenvalues $\pm\theta$ are spanned by*

$$\begin{pmatrix} \mathbf{e}_i \\ \frac{1}{\theta^2-\lambda^2}(A_{\dot{H}} \pm \theta I)\mathbf{b}_i^\intercal \end{pmatrix},\ 1 \leq i \leq t,$$

or

$$\begin{pmatrix} B^\intercal \mathbf{x}_i \\ \frac{1}{\theta^2-\lambda^2}(A_{\dot{H}} \pm \theta I)BB^\intercal \mathbf{x}_i \end{pmatrix},\ 1 \leq i \leq t.$$

Proof. The result follows from Equation (6) and the previous theorem. □

The r-dimensional cube Q_r is an r-regular graph with $2r$ vertices whose vertex set consists of all binary r-tuples. Vertices are joined by an edge precisely if they coincide in $r-1$ coordinates. This cube can be viewed as the underlying graph of a set of r-dimensional signed cubes. In particular, we know from (Stanić I, 2019) that there is the unique r-dimensional signed cube with negative quadrangles; when say unique we mean up to switching isomorphism. Its spectrum is $[\sqrt{r}, -\sqrt{r}]$ and, with possible relabelling of the vertices, its adjacency matrix is

$$A_{\dot{Q}_r} = \begin{pmatrix} O & N^\intercal & I & O \\ N & O & O & -I \\ I & O & O & N^\intercal \\ O & -I & N & O \end{pmatrix},$$

where the 2×2 top-left block is $A_{\dot{Q}_{r-1}}$.

Corollary 4.3. *If $\mathbf{x}_1 = \begin{pmatrix} \mathbf{x}_{11} \\ \mathbf{x}_{12} \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} \mathbf{x}_{21} \\ \mathbf{x}_{22} \end{pmatrix}, \dots, \mathbf{x}_{r-1} = \begin{pmatrix} \mathbf{x}_{r-1,1} \\ \mathbf{x}_{r-1,2} \end{pmatrix}$ is the full set of linearly independent eigenvectors of the $(r-1)$-dimensional cube with negative quadrangles $\dot{Q}_{r-1}$, where $\mathbf{x}_{i1}$ and $\mathbf{x}_{i2}$ are equal in length, then the eigenspaces of $\pm\sqrt{r}$ of $\dot{Q}_r$ are spanned by*

$$\begin{pmatrix} \mathbf{e}_i \\ (A_{\dot{H}} \pm \sqrt{r} I)\mathbf{b}_i \end{pmatrix},\ 1 \leq i \leq r-1,$$

or

$$\begin{pmatrix} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \\ (\sqrt{r-1} \pm \sqrt{r})\mathbf{x}_i \end{pmatrix},\ 1 \leq i \leq r-1.$$

Proof. Since $B = B^\intercal = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$ and $(\pm\sqrt{r}I - A_{\dot{H}})^{-1} = A_H \pm \sqrt{r}I$, the result follows from Corollary 4.2. □

Remark 4.4. A *weighing matrix* M of weight r is a $(0, 1, -1)$-matrix satisfying $M^\intercal M = rI$. Throughout the paper we established constructions of certain families of signed graphs having 2 symmetric eigenvalues. We note that adjacency matrices of these signed graphs can be recognized as weighing matrices whose weights are equal to the vertex degree in the corresponding signed graph. Weighing matrices are considered in (Harada and Munemasa, 2012) and some therein references.

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