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A class of linear and nonlinear Fredholm integral equations of the third kind

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Abstract

In this paper we are applying a new approach to prove the uniqueness and existence theorems for linear and nonlinear Fredholm integral equations of the third kind.

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1. Introduction

Various issues concerning the theory of integral equations were studied by Lavrent'ev (1959); Magnitskii (1979); Lavrent'ev *et al.* (1986); Imanaliev & Asanov (1989; 2007, 2010); Asanov (1998); Bukhgeim (1999); Denisov (1999); Shishatskii *et al*. (2001); Apartsyn (2003); Imanaliev *et al*. (2011); Ismat Beg *et al*. (2014). More specifically, regularizing operators in the sense of Lavrent'ev were constructed by Lavrent'ev (1959) for solving linear Fredholm integral equations of the first kind. Imanaliev & Asanov (2007, 2010) and Imanaliev *et al.* (2011) proved uniqueness theorems for systems of nonlinear Volterra integral equations of the third kind and for systems of linear Fredholm integral equations of the third kind and constructed regularizing operators in the sense of Lavrent'ev. In this paper a new approach is proposed for the study of Fredholm integral equations of the third kind. Following this approach, we prove uniqueness and existence theorems for the linear and nonlinear Fredholm integral equation of the third kind.

2. Preliminaries

Let consider the linear and nonlinear integral equations of the third kind

$$P(x)u(x)=\lambda\int_a^b K(x,y)u(y)dy+f(x)\ ,\ x\in[a,b]\,, \quad (1)$$

$$P(x)v(x)=\lambda\int_a^b M(x,y,v(y))dy+f(x)\ ,\ x\in[a,b]\,, \quad (2)$$

where $P(x)$ and $f(x)$ are given continuous functions on $[a,b]$, $K(x,y)$ is given continuous function on $G=[a,b]^2$, $M(x,y,v)$ is given continuous function on $G\times R$, $u(x)$ and $v(x)$ are sought continuous functions on $[a,b]$, λ is a real parameter, $P(x_i)=0$, $x_i\in[a,b]$, $i=1,2,\dots,m$.

Throughout this paper we assume that

$$P(x)=\prod_{i=1}^{m}\prod_{j=1}^{s(i)}P_{i,j}(x),\quad P_{i,j}(x_i)=0,\quad P_{i,j}(x)\in C[a,b]$$

$$P_{i,j}(x)\neq 0,\quad x\neq x_i,\quad i=1,\dots,m,\quad j=1,\dots,s(i)\,. \quad (3)$$

Setting $x=x_1$, we find from (1) and (2) that

$$\lambda\int_a^b K(x_1,y)u(y)dy+f(x_1)=0\,, \quad (4)$$

$$\lambda\int_a^b M(x_1,y,v(y))dy+f(x_1)=0\,. \tag{5}$$

Subtracting (4) from (1) and (5) from (2) yield

$$P(x)u(x)=\lambda\int_a^b\left[K(x,y)-K(x_1,y)\right]u(y)dy+f(x)-f(x_1)\,, \tag{6}$$

$$P(x)v(x)=\lambda\int_a^b\left[M(x,y,v(y))-M(x_1,y,v(y))\right]dy+f(x)-f(x_1)\,, \tag{7}$$

where $x\in[a,b]$.

Assume that the following conditions hold:

a) For all $i=1,...,m$, $j=s_{i-1}+1,\ ...,\ s_i$, $K_{i,j}(x,y)\in C(G)$,

where

$$s_0=0,\quad s_i=\sum_{j=1}^{i}s(j),\quad K_{1,0}(x,y)=K(x,y),$$

$$K_{1,j_1}(x,y)=\frac{1}{P_{1,j_1}(x)}\left[K_{1,j_1-1}(x,y)-K_{1,j_1-1}(x_1,y)\right]\ \left(j_1=1,...,s_1\right),$$

$$K_{2,s_1}(x,y)=K_{1,s_1}(x,y)\,,\quad K_{2,j_2}(x,y)=\frac{1}{P_{2,j_2-s_1}(x)}\left[K_{2,j_2-1}(x,y)-K_{2,j_2-1}(x_2,y)\right]$$

$$\left(j_2=s_1+1,...,s_2\right),\ \dots,\ K_{m,s_{m-1}}(x,y)=K_{m-1,s_{m-1}}(x,y)\,,$$

$$K_{m,j_m}(x,y)=\frac{1}{P_{m,j_m-s_{m-1}}(x)}\left[K_{m,j_m-1}(x,y)-K_{m,j_m-1}(x_m,y)\right]\ \left(j_m=s_{m-1}+1,...,s_m\right);$$

b) For all $i=1,...,m$, $j=s_{i-1}+1,...,s_i$, $M_{i,j}(x,y,v)\in C(G\times R)$, where

$$s_0=0,\quad s_i=\sum_{j=1}^{i}s(j),\quad M_{1,0}(x,y,v)=M(x,y,v),$$

$$M_{1,j_1}(x,y,v)=\frac{1}{P_{1,j_1}(x)}\left[M_{1,j_1-1}(x,y,v)-M_{1,j_1-1}(x_1,y,v)\right]\ \left(j_1=1,...,s_1\right),$$

$$M_{2,s_1}(x,y,v)=M_{1,s_1}(x,y,v)\,,$$

$$M_{2,j_2}(x,y,v)=\frac{1}{P_{2,j_2-s_1}(x)}\left[M_{2,j_2-1}(x,y,v)-M_{2,j_2-1}(x_2,y,v)\right]$$

$$\left(j_2=s_1+1,...,s_2\right),\ \dots,\ M_{m,s_{m-1}}(x,y,v)=M_{m-1,s_{m-1}}(x,y,v)\,,$$

$$M_{m,j_m}(x,y,v)=\frac{1}{P_{m,j_m-s_{m-1}}(x)}\left[M_{m,j_m-1}(x,y,v)-M_{m,j_m-1}(x_m,y,v)\right]\ \left(j_m=s_{m-1}+1,...,s_m\right);$$

c) For all $i = 1,..., m, j = s_{i-1}+ 1, ... , s_i$, $F_{i,j}(x) \in C[a,b]$, where

$$s_0 = 0, \quad s_i = \sum_{j=1}^{i} s(j), \quad F_{1,0}(x) = f(x),$$

$$F_{1,j_1}(x) = \frac{1}{P_{1,j_1}(x)}\left[F_{1,j_1-1}(x) - F_{1,j_1-1}(x_1)\right] \ \left(j_1 = 1,...,s_1\right), \ F_{2,s_1}(x) = F_{1,s_1}(x),$$

$$F_{2,j_2}(x) = \frac{1}{P_{2,j_2-s_1}(x)}\left[F_{2,j_2-1}(x) - F_{2,j_2-1}(x_2)\right]\left(j_2 = s_1+1,...,s_2\right), \ ...,$$

$$F_{m,s_{m-1}}(x) = F_{m-1,s_{m-1}}(x), \quad F_{m,j_m}(x) = \frac{1}{P_{m,j_m-s_{m-1}}(x)}\left[F_{m,j_m-1}(x) - F_{m,j_m-1}(x_m)\right],$$

$$\left(j_m = s_{m-1}+1,...,s_m\right).$$

3. The linear Fredholm integral equation of the third kind

Theorem 1. Let conditions (3), a) and c) are satisfied. Then the solution of the integral equation (1) in $C[a,b]$ is equivalent to the solution of the linear integral equations of the second kind

$$u(x) = \lambda\int_a^b K_{m,s_m}(x,y)u(y)dy + F_{m,s_m}(x)\ , \ x \in [a,b], \tag{8}$$

with the conditions

$$\begin{cases} \lambda\int_a^b K_{1,j_1-1}(x_1,y)u(y)dy + F_{1,j_1-1}(x_1) = 0, \ j_1 = 1,...,s_1\ , \\ \lambda\int_a^b K_{2,j_2-1}(x_2,y)u(y)dy + F_{2,j_2-1}(x_2) = 0, \ j_2 = s_1+1,...,s_2\ , \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \lambda\int_a^b K_{m,j_m-1}(x_m,y)u(y)dy + F_{m,j_m-1}(x_m) = 0, \ j_m = s_{m-1}+1,...,s_m. \end{cases} \tag{9}$$

Proof. First, let $u(x) \in C[a,b]$ is a solution of equation (1). Then identities (4) and (6) hold. Taking into account (3) and conditions a) and c) we find from (6) that

$$\left(\prod_{j_1=2}^{s(1)} P_{1,j_1}(x)\right)\left(\prod_{i=2}^{m}\prod_{j=1}^{s(i)} P_{i,j}(x)\right)u(x) = \lambda\int_a^b K_{1,1}(x,y)u(y)dy + F_{1,1}(x)\ , \ x \in [a,b]. \tag{10}$$

If $P(x)=P_{1,1}(x)$, $x \in [a,b]$, then

$$\left(\prod_{j_1=2}^{s(1)} P_{1,j_1}(x)\right)\left(\prod_{i=2}^{m}\prod_{j=1}^{s(i)} P_{i,j}(x)\right) = 1\ , \quad x \in [a,b].$$

If $s_1 = s(1) = 1$ and $P(x_2) = 0$, then

$$\left(\prod_{j_1=2}^{s(1)} P_{1,j_1}(x)\right) = 1\ , \quad x \in [a,b].$$

If $s_1 > 1$, then setting $x = x_1$, we find from (10) that

$$\lambda\int_a^b K_{1,1}(x_1,y)u(y)dy + F_{1,1}(x_1) = 0\,. \tag{11}$$

Subtracting (11) from (10) and taking into account conditions (3) , a) and c), yields

$$\left(\prod_{j_1=3}^{s(1)} P_{1,j_1}(x)\right)\left(\prod_{i=2}^{m}\prod_{j=1}^{s(i)} P_{i,j}(x)\right)u(x) = \lambda\int_a^b K_{1,2}(x,y)u(y)dy + F_{1,2}(x)\,,\ x\in[a,b]\,.$$

If $s_1 = 2$, then

$$\left(\prod_{j_1=3}^{s(1)} P_{1,j_1}(x)\right) = 1\,,\quad x\in[a,b]\,.$$

Continuing this process, we see that the function $u(x)$ solves the following equation

$$\left(\prod_{i=2}^{m}\prod_{j=1}^{s(i)} P_{i,j}(x)\right)u(x) = \lambda\int_a^b K_{2,s_1}(x,y)u(y)dy + F_{2,s_1}(x) \tag{12}$$

with the conditions

$$\lambda\int_a^b K_{1,j_1-1}(x_1,y)u(y)dy + F_{1,j_1-1}(x_1) = 0\,,\ \ j_1 = 1,\ldots,s_1\,.$$

Setting $x = x_2$ we find from (12) that

$$\lambda\int_a^b K_{2,s_1}(x_2,y)u(y)dy + F_{2,s_1}(x_2) = 0\,. \tag{13}$$

Subtracting (13) from (12) and taking into account conditions a) and c), yields

$$\left(\prod_{j_2=2}^{s(2)} P_{2,j_2}(x)\right)\left(\prod_{i=3}^{m}\prod_{j=1}^{s(i)} P_{i,j}(x)\right)u(x) = \lambda\int_a^b K_{2,s_1+1}(x,y)u(y)dy + F_{2,s_1+1}(x)\,,\ x\in[a,b]\,. \tag{14}$$

Continuing this process with respect to equation (14), we see that the function $u(x)$ solves equation (8) with the conditions (9).

Conversely, let $u(x)\in C[a,b]$ is a solution of the equation (8) with conditions (9). Multiplying the equation (8) by $P_{m,s(m)}(x)$ and taking into account condition (9) at $j_m = s_m$, gives

$$P_{m,s(m)}(x)u(x) = \lambda\int_a^b K_{m,s_m-1}(x,y)u(y)dy + F_{m,s_m-1}(x)\,,\ \ x\in[a,b]\,. \tag{15}$$

Multiplying the equation (15) by $P_{m,s(m)-1}(x)$ and taking into account condition (9) at $j_m = s_{m-1}$, we have

$$P_{m,s(m)-1}(x)P_{m,s(m)}u(x) = \lambda\int_a^b K_{m,s_m-2}(x,y)u(y)dy + F_{m,s_m-2}(x)\,,\ \ x\in[a,b]\,. \tag{16}$$

Continuing this process with respect to equation (16) and taking into account condition (9), we see that $u(x)$ is a solution of the equation (1). The theorem 1 is proved.

Corollary 1. Let conditions (3), a) and c) are satisfied and $\frac{1}{\lambda}$ is a real number, that is not an eigenvalue of kernel $K_{m,s_m}(x,y)$. Then:

1) The solution of equation (1) is unique in $C[a,b]$.

2) The solution of equation (8) can be written as

$$u(x) = F_{m,s_m}(x) + \int_a^b R(x,y,\lambda)F_{m,s_m}(y)dy\,,\ \ x\in[a,b]\,. \tag{17}$$

where $R(x,y,\lambda)$ is the resolvent of the kernel $\lambda K_{m,s_m}(x,y)$. In this case, the function $u(x)$, defined by (17) is a solution of the equation (1) if and only if $u(x)$ satisfies the condition (9).

Corollary 2. Let conditions (3), a) and c) are satisfied, $\frac{1}{\lambda}$ is a real number, that is an eigenvalue of the kernel $K_{m,s_m}(x,y)$ and the functions $\varphi_1(x),\varphi_2(x),...,\varphi_q(x)$ and $\psi_1(x)$, $\psi_2(x)$, ..., $\psi_q(x)$ are the eigenfunctions of the kernels $K_{m,s_m}(x,y)$ and $K_{m,s_m}(y,x)$ that correspond to the eigenvalue $\frac{1}{\lambda}$. Then the following assertions hold:

1) If there exists $i\in\{1, 2,..., q\}$, such that

$$\int_a^b F_{m,s_m}(x)\psi_i(x)dx \neq 0,$$

then the equation (1) has no solution in $C[a,b]$.

2) If

$$\int_a^b F_{m,s_m}(x)\psi_i(x)dx = 0$$

for all i = 1,2,..., q and $r(A) \neq r(B)$, where A is an $s_m\times q$ matrix with $s_i = \sum_{j=1}^{i} s(j)$, i = 1, 2, ...,m,

$$A=\begin{pmatrix} A_1\\ A_2\\ \vdots\\ A_m\end{pmatrix},\ Q=\begin{pmatrix} B_1\\ B_2\\ \vdots\\ B_m\end{pmatrix},\ B=(A,Q), \tag{18}$$

$$A_1=\begin{pmatrix} a_{11,1} & \dots & a_{11,q}\\ \dots & \dots & \dots\\ a_{1s_1,1} & \dots & a_{1s_1,q}\end{pmatrix},\ B_1=\begin{pmatrix} b_{11}\\ \vdots\\ b_{1s_1}\end{pmatrix},$$

$$A_i=\begin{pmatrix} a_{is_{i-1}+1,1} & \dots & a_{is_{i-1}+1,q}\\ \dots & \dots & \dots\\ a_{is_i,1} & \dots & a_{is_i,q}\end{pmatrix},\quad B_i=\begin{pmatrix} b_{is_{i-1}+1}\\ \vdots\\ b_{is_i}\end{pmatrix},$$

$$a_{1j_1,p} = \lambda\int_a^b K_{1,j_1-1}(x_1,y)\varphi_p(y)dy,$$

$$b_{1j_1} = -F_{1,j_1-1}(x_1) - \lambda\int_a^b K_{1,j_1-1}(x_1,y)\varphi_0(y)dy,\ j_1 = 1,...,s_1,$$

$$a_{ij_i,p} = \lambda\int_a^b K_{i,j_i-1}(x_i,y)\varphi_p(y)dy,$$

$$b_{ij_i} = -F_{j,j_i-1}(x_i) - \lambda\int_a^b K_{i,j_i-1}(x_i,y)\varphi_0(y)dy,$$

$$j_i = s_{i-1}+1,...,s_i,\ i = 2,...,m,\ p = 1,...,q,$$

$r(A)$ is the rank of the matrix A and $\varphi_0(x)$ is a particular solution of the equation (8), then equation (1) has no solutions in $C[a,b]$.

3) If

$$\int_a^b F_{m,s_m}(x)\psi_i(x)dx = 0$$

for all i = 1, ..., q and $r(A) = r(B) = q$, then equation (1) has a unique solution in $C[a,b]$ and that solution can be represented as

$$u(x) = \varphi_0(x) + \sum_{j=1}^{q} c_j\varphi_j(x),\ x\in[a,b]. \tag{19}$$

Here $c = (c_1, c_2, \dots, c_q)^T$ is the only vector satisfying the system

$$Ac = Q, \tag{20}$$

where the matrices A and Q are defined by formula (18).

4) If

$$\int_a^b F_{m,s_m}(x)\psi_i(x)dx = 0$$

for all i = 1, 2, ..., q and $r = r(A) = r(B) < q$, then equation (1) has a solution in $C[a, b]$ and that solution can be represented as (19), where the vector $c = (c_1, c_2, \dots, c_q)^T$ depends on q-r arbitrary constants and satisfies system (20).

Proof. In case 1) by the Fredholm alternative the equation (8) has no solution in $C[a,b]$. Therefore, the equation (1) has no solution in $C[a,b]$ either. In cases 2)-4), by the Fredholm alternative the equation (8) has a solution representable as (19), where $c_1, c_2, \dots, c_q$ are arbitrary constants. Substituting (19) into (9) we have system (20). Applying the Kronecker-Capelli's theorem to system (20), we prove assertions 2)-4) in Corollary 2.

Example 1. Consider the equation

$$x^2\left(x-\frac{1}{2}\right)^2(x-1)u(x) = \lambda\int_0^1\left(1+2x^6y-4x^4y^2\right)u(y)dy +$$

$$+\alpha x^6 + \alpha_1x^4 + \beta x^3 + \gamma_1x^2 + \gamma_2,\ x\in[0,1], \tag{21}$$

where λ, α, α_1, β, γ_1, γ_2 are real parameters. In this case

$$x_1 = 0\,,\; x_2 = \frac{1}{2}\,,\; x_3 = 1\,,\; m = 3\,,\; s(1) = 1\,,\; s(2) = 2\,,\; s(3) = 1\,,$$

$$P_{1,1}(x) = x^2\,,\; P_{2,1}(x) = P_{2,2}(x) = x - \frac{1}{2}\,,\; P_{3,1}(x) = x - 1\,,$$

$$K_{1,1}(x,y) = 2x^4 y - 4x^2 y^2\,,\; K_{2,2}(x,y) = (2x+1)\left(x^2 + \frac{1}{4}\right)y - 4\left(x + \frac{1}{2}\right)y^2\,,$$

$$K_{2,3}(x,y) = 2\left(x^2 + x + \frac{3}{4}\right)y - 4y^2\,,\; K_{3,4}(x,y) = 2(x+2)y\,,$$

$$F_{1,1}(x) = \alpha x^4 + \alpha_1 x^2 + \beta x + \gamma_1\,,\; F_{2,2}(x) = \alpha\left(x + \frac{1}{2}\right)\left(x^2 + \frac{1}{4}\right) + \alpha_1\left(x + \frac{1}{2}\right) + \beta\,,$$

$$F_{2,3}(x) = \alpha\left(x^2 + x + \frac{3}{2}\right) + \alpha_1\,,\; F_{3,4}(x) = \alpha(x+2)\,.$$

Then by theorem 1, the solution of the integral equation (21) in $C[0,1]$ is equivalent to the solution of the integral equation of the second kind

$$u(x) = 2\lambda(x+2)\int_0^1 y u(y)dy + (x+2)\alpha\,,\; x \in [0,1] \tag{22}$$

with the conditions

$$\begin{cases} \lambda\int_0^1 u(y)dy + \gamma_2 = 0, \\ \lambda\int_0^1 \left(\frac{1}{8}y - y^2\right)u(y)dy + \frac{1}{16}\alpha + \frac{1}{4}\alpha_1 + \frac{1}{2}\beta + \gamma_1 = 0, \\ \lambda\int_0^1 \left(y - 4y^2\right)u(y)dy + \frac{1}{2}\alpha + \alpha_1 + \beta = 0, \\ \lambda\int_0^1 \left(\frac{11}{2}y - 4y^2\right)u(y)dy + \frac{11}{4}\alpha + \alpha_1 = 0. \end{cases} \tag{23}$$

Then from (22) and (23) we obtain:

1) Let $\lambda \neq \frac{3}{8}$. Then the equation (21) has a solution in $C[0,1]$ if and only if

$$\alpha_1 = \frac{11\alpha(4\lambda - 3)}{4(3 - 8\lambda)}\,,\; \beta = \frac{27\alpha}{4(3 - 8\lambda)}\; \gamma_1 = -\frac{3\alpha}{2(3 - 8\lambda)}\,,\; \gamma_2 = -\frac{15\alpha\lambda}{2(3 - 8\lambda)}\,.$$

This solution is unique and given by the formula

$$u(x) = \frac{3\alpha(x+2)}{3 - 8\lambda}\,,\; x \in [0,1]\,. \tag{24}$$

2) Let $\lambda = \frac{3}{8}$ and $\alpha \neq 0$. Then the equation (21) has no solution in $C[0,1]$.

3) Let $\lambda = \frac{3}{8}$ and $\alpha = 0$. Then the equation (21) has a solution in $C[0,1]$$ if and only if

$$\beta = -\frac{18}{11}\alpha_1\,,\; \gamma_1 = \frac{4}{11}\alpha_1\,,\; \gamma_2 = \frac{15}{22}\alpha_1\,.$$

This solution is unique and given by the formula

$$u(x) = -\frac{8}{11}\alpha_1\left(x+2\right),\ x \in [0,1]. \tag{25}$$

Example 2. Consider the equation

$$\ln(1+x)\cos\left(x+\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}-x\right)u(x) = \lambda\int_0^{\frac{\pi}{2}}\left\{\cos y+\sin\left(\frac{\pi}{2}-x\right)\times\right.$$

$$\left.\times\left[\sin y+\ln(1+x)\left(\left(\sin x-\frac{8}{\pi^2}y\right)\cos\left(x+\frac{\pi}{4}\right)+y\right)\right]\right\}u(y)dy+ \tag{26}$$

$$+\alpha_0+\sin\left(\frac{\pi}{2}-x\right)\left\{\alpha_1+\ln(1+x)\left[\alpha_2\cos\left(x+\frac{\pi}{4}\right)+\alpha_3\right]\right\},\ x\in\left[0,\frac{\pi}{2}\right],$$

where $\lambda, \alpha_0, \alpha_1, \alpha_2, \alpha_3$ are real parameters. In this case

$$x_1 = \frac{\pi}{2},\ x_2 = 0,\ x_3 = \frac{\pi}{4},\ m = 3,\ s(1) = 1,\ s(2) = 1,\ s(3) = 1,$$

$$P_{1,1}(x) = \sin\left(\frac{\pi}{2}-x\right),\ P_{2,1}(x) = \ln(1+x),\ P_{3,1}(x) = \cos\left(x+\frac{\pi}{4}\right),$$

$$K_{1,1}(x,y) = \sin y+\ln(1+x)\left[\left(\sin x-\frac{8}{\pi^2}y\right)\cos\left(x+\frac{\pi}{4}\right)+y\right],$$

$$K_{1,2}(x,y) = \left(\sin x-\frac{8}{\pi^2}y\right)\cos\left(x+\frac{\pi}{4}\right)+y,\ K_{2,3}(x,y) = \sin x-\frac{8}{\pi^2}y,$$

$$F_{1,1}(x) = \alpha_1+\ln\left(1+x\right)\left[\alpha_2\cos\left(x+\frac{\pi}{4}\right)+\alpha_3\right],\ F_{1,2}(x) = \alpha_2\cos\left(x+\frac{\pi}{4}\right)+\alpha_3,\ F_{2,3}(x) = \alpha_2.$$

Then by theorem 1, the solution of the integral equation (26) in $C\left[0,\frac{\pi}{2}\right]$ is equivalent to the solution of the integral equation of the second kind

$$u(x) = \lambda\int_0^{\frac{\pi}{2}}\left(\sin x-\frac{8}{\pi^2}y\right)u(y)dy+\alpha_2,\ x\in\left[0,\frac{\pi}{2}\right] \tag{27}$$

with the conditions

$$\begin{cases}\lambda\int_0^{\frac{\pi}{2}}u(y)\cos y\,dy+\alpha_0 = 0,\\ \lambda\int_0^{\frac{\pi}{2}}u(y)\sin y\,dy+\alpha_1 = 0,\\ \lambda\int_0^{\frac{\pi}{2}}yu(y)dy+\alpha_3 = 0.\end{cases} \tag{28}$$

Then from (27) and (28) we obtain:

The equation (26) has a solution in $C\left[0,\frac{\pi}{2}\right]$ if and only if

$$\alpha_0=\frac{\pi\lambda^2\alpha_2}{4\Delta(\lambda)}-\frac{\lambda(\lambda-1)\alpha_2}{\Delta(\lambda)},\ \alpha_1=\frac{\pi^2\lambda^2\alpha_2}{8\Delta(\lambda)}-\frac{\lambda(\lambda-1)\alpha_2}{\Delta(\lambda)},$$

$$\alpha_3=\frac{\pi\lambda^2\alpha_2}{2\Delta(\lambda)}-\frac{\pi^2\lambda(\lambda-1)\alpha_2}{8\Delta(\lambda)},$$

where

$$\Delta(\lambda)=-\left[\left(\frac{4}{\pi}-1\right)\lambda^2+1\right]\le -1\quad \forall\lambda\in R.$$

This solution is unique and given by the formula

$$u(x)=-\frac{\pi\lambda}{2\Delta(\lambda)}\alpha_2\sin x+\frac{(\lambda-1)\alpha_2}{\Delta(\lambda)},\ x\in\left[0,\frac{\pi}{2}\right]. \quad (29)$$

4. The nonlinear Fredholm integral equation of the third kind

Theorem 2. Let conditions (3), b) and c) are satisfied. Then the solution of the nonlinear integral equation (2) in $C[a,b]$ is equivalent to the solution of the integral equation of the second kind

$$v(x)=\lambda\int_a^b M_{m,s_m}(x,y,v(y))dy+F_{m,s_m}(x),\ x\in[a,b] \quad (30)$$

with the conditions

$$\begin{cases}\lambda\int_a^b M_{1,j_1-1}(x_1,y,v(y))dy+F_{1,j_1-1}(x_1)=0,\ j_1=1,\dots,s_1,\\ \lambda\int_a^b M_{2,j_2-1}(x_2,y,v(y))dy+F_{2,j_2-1}(x_2)=0,\ j_2=s_1+1,\dots,s_2,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \lambda\int_a^b M_{m,j_m-1}(x_m,y,v(y))dy+F_{m,j_m-1}(x_m)=0,\ j_m=s_{m-1}+1,\dots,s_m.\end{cases} \quad (31)$$

Proof. First, let $u(t)\in C[a,b]$ is a solution of the equation (2). Then identities (5) and (7) hold. Taking into account (3) and conditions b) and c) we find from (7) that

$$\left(\prod_{j_1=2}^{s(1)}P_{1,j_1}(x)\right)\left(\prod_{i=2}^{m}\prod_{j=1}^{s(i)}P_{i,j}(x)\right)v(x)=\lambda\int_a^b M_{1,1}(x,y,v(y))dy+F_{1,1}(x),\ x\in[a,b]. \quad (32)$$

If $P(x)=P_{1,1}(x)$, $x\in[a,b]$, then

$$\left(\prod_{j_1=2}^{s(1)}P_{1,j_1}(x)\right)\left(\prod_{i=2}^{m}\prod_{j=1}^{s(i)}P_{i,j}(x)\right)=1,\quad x\in[a,b].$$

If $s_1=s(1)=1$ and $P(x_2)=0$, then

$$\left(\prod_{j_1=2}^{s(1)}P_{1,j_1}(x)\right)=1,\quad x\in[a,b].$$

If $s_1>1$, then setting $x=x_1$, we find from (32) that

$$\lambda\int_a^b M_{1,1}(x_1,y,v(y))dy+F_{1,1}(x_1)=0. \quad (33)$$

Subtracting (33) from (32) and taking into account conditions (3) , b) and c), yields

$$\left(\prod_{j_1=3}^{s(1)}P_{1,j_1}(x)\right)\left(\prod_{i=2}^{m}\prod_{j=1}^{s(i)}P_{i,j}(x)\right)v(x)=\lambda\int_a^b M_{1,2}(x,y,v(y))dy+F_{1,2}(x),\ x\in[a,b].$$

If $s_1 = 2$, then

$$\left(\prod_{j_1=3}^{s(1)} P_{1,j_1}(x)\right) = 1, \quad x \in [a,b].$$

Continuing this process, we see that the function $v(x)$ solves the following equation

$$\prod_{i=2}^{m}\prod_{j=1}^{s(i)} P_{i,j}(x)v(x) = \lambda\int_a^b M_{2,s_1}(x,y,v(y))dy + F_{2,s_1}(x) \quad (34)$$

with the conditions

$$\lambda\int_a^b M_{1,j_1-1}(x_1,y,v(y))dy + F_{1,j_1-1}(x_1) = 0, \quad j_1 = 1,...,s_1.$$

Setting $x = x_2$ we find from (34) that

$$\lambda\int_a^b M_{2,s_1}(x_2,y,v(y))dy + F_{2,s_1}(x_2) = 0. \qquad (35)$$

Subtracting (35) from (34) and taking into account conditions b) and c), yields

$$\left(\prod_{j_2=2}^{s(2)} P_{2,j_2}(x)\right)\left(\prod_{i=3}^{m}\prod_{j=1}^{s(i)} P_{i,j}(x)\right)v(x) =$$

$$= \lambda\int_a^b M_{2,s_1+1}(x,y,v(y))dy + F_{2,s_1+1}(x), \ x \in [a,b]. \quad (36)$$

Continuing this process with respect to equation (36), we see that the function $v(x)$ solves equation (30) with the conditions (31).

Conversely, let $v(x) \in C[a,b]$ is a solution of the equation (30) with conditions (31). Multiplying the equation (30) by $P_{m,s(m)}(x)$ and taking into account condition (31) at $j_m = s_{m-1}$, gives

$$P_{m,s(m)}(x)v(x) = \lambda\int_a^b M_{m,s_m-1}(x,y,v(y))dy + F_{m,s_m-1}(x), \ x \in [a,b]. \qquad (37)$$

Multiplying the equation (37) by $P_{m,s(m)-1}(x)$ and taking into account condition (31) at $j_m = s_{m-1}$, we have

$$P_{m,s(m)-1}(x)P_{m,s(m)}v(x) = \lambda\int_a^b M_{m,s_m-2}(x,y,v(y))dy + F_{m,s_m-2}(x), \ x \in [a,b]. \qquad (38)$$

Continuing this process with respect to equation (38) and taking into account conditions (31), we see that $v(x)$ is a solution of the equation (2). The theorem 2 is proved.

Corollary 3. Let conditions (2) and b) are satisfied. But the condition c) is not satisfied. Then the equation (2) has no solution on $C[a,b]$.

Example 3. Consider the equation

$$x\left(x-\frac{1}{2}\right)(x-1)v(x) = \int_0^1\left[\left(\sqrt{x}+x^3y\right)v(y)+y^2v^3(y)\right]dy +$$

$$+\alpha_0x^4+\alpha x^3+\alpha_1x^2+\beta x+\beta_1\sqrt{x}+\gamma, \ x \in [0,1], \qquad (39)$$

where $\alpha_0, \alpha, \alpha_1, \beta, \beta_1$ and γ are real parameters. It is easy to see that equation (39) satisfies conditions (3), b) and c) for

$$m = 3, s(1) = 2, \lambda = 1, \ s(2) = s(3) = 1, \ P_{1,1}(x) = P_{1,2}(x) = \sqrt{x}, \ P_{2,1}(x) = x - \frac{1}{2},$$

$$P_{3,1}(x) = x-1, \ M_{1,0}(x,y,v) = M(x,y,v) = \left(\sqrt{x}+x^3y\right)v + y^2v^3,$$

$$M_{1,1}(x,y,v) = \left(1+x^2\sqrt{x}y\right)v, \ M_{1,2}(x,y,v) = M_{2,2}(x,y,v) = x^2yv,$$

$$M_{2,3}(x,y,v) = M_{3,3}(x,y,v) = \left(x+\frac{1}{2}\right)yv, \ M_{3,4}(x,y,v) = yv,$$

$$F_{1,0}(x)=f(x)=\alpha_0x^4+\alpha x^3+\alpha_1x^2+\beta x+\beta_1\sqrt{x}+\gamma\,,$$

$$F_{1,1}(x)=\alpha_0x^3\sqrt{x}+\alpha x^2\sqrt{x}+\alpha_1x\sqrt{x}+\beta\sqrt{x}+\beta_1\,,$$

$$F_{2,2}(x)=F_{1,2}(x)=\alpha_0x^3+\alpha x^2+\alpha_1x+\beta\,,$$

$$F_{3,3}(x)=F_{2,3}(x)=\alpha_0\left(x^2+\frac{1}{2}x+\frac{1}{4}\right)+\alpha\left(x+\frac{1}{2}\right)+\alpha_1\,,\ F_{3,4}(x)=\alpha_0x+\frac{3}{2}\alpha_0+\alpha\,.$$

Then, for equation (39), the equation (30) and conditions (31) are written as

$$v(x)=\int_0^1 yv(y)dy+\alpha_0x+\frac{3}{2}\alpha_0+\alpha\,,\ x\in[0,1]$$

with the conditions

$$\begin{cases}\int_0^1 y^2v^3(y)dy+\gamma=0,\\ \int_0^1 v(y)dy+\beta_1=0,\\ \int_0^1\frac{1}{4}yv(y)dy+\frac{1}{8}\alpha_0+\frac{1}{4}\alpha+\frac{1}{2}\alpha_1+\beta=0,\\ \int_0^1\frac{3}{2}yv(y)dy+\frac{7}{4}\alpha_0+\frac{3}{2}\alpha+\alpha_1=0.\end{cases}$$

Then the equation (39) has a solution in $C[0,1]$ if and only if

$$\alpha_1=-5\alpha_0-3\alpha\,,\ \beta=\frac{11}{6}\alpha_0+\alpha\,,\ \beta_1=-\frac{25}{6}\alpha_0-2\alpha\,,$$

$$\gamma=-\left[\frac{\alpha_0^3}{6}+\frac{3}{5}\alpha_0^2\left(\frac{11}{3}\alpha_0+2\alpha\right)+\frac{3}{4}\alpha_0\left(\frac{11}{3}\alpha_0+2\alpha\right)^2+\frac{1}{3}\left(\frac{11}{3}\alpha_0+2\alpha\right)^3\right].$$

This solution is unique and given by the formula

$$v(x)=\alpha_0x+\frac{11}{3}\alpha_0+2\alpha\,,\ x\in[0,1]\,.$$

Example 4. Consider the equation

$$\cos x\left(e^{x-1}-1\right)v(x)=\lambda\int_0^{\frac{\pi}{2}}\left\{e^{v(y)}y+\cos x\left[v^2(y)\left(e^{x-1}-1\right)+v^4(y)\right]\right\}dy+\alpha_0+\alpha_1\cos x\,,\ x\in\left[0,\frac{\pi}{2}\right], \qquad (40)$$

where $\lambda,\alpha_0,\alpha_1$ are real parameters, $\lambda\neq0$. In this case

$$x_1=\frac{\pi}{2}\,,\ x_2=1\,,\ m=2,\ s(1)=1,\ s(2)=1\,,$$

$$P_{1,1}(x)=\cos x\,,\ P_{2,1}(x)=e^{x-1}-1\,,\ M_{1,1}(x,y,v)=v^2\left(e^{x-1}-1\right)+v^4\,,$$

$$M_{2,2}(x,y,v)=v^2\,,\ F_{1,1}(x)=\alpha_1\,,\ F_{2,2}(x)=0\,.$$

Then, for equation (40) the equation (30) and conditions (31) are written as

$$v(x) = \lambda \int_0^{\frac{\pi}{2}} v^2(s)ds \tag{41}$$

with the conditions

$$\begin{cases} \lambda \int_0^{\frac{\pi}{2}} e^{v(y)} y dy + \alpha_0 = 0, \\ \lambda \int_0^{\frac{\pi}{2}} v^4(y)dy + \alpha_1 = 0. \end{cases} \tag{42}$$

Then from (41) and (42) we obtain:

1) The function $v(x) = 0,\ x \in \left[0, \frac{\pi}{2}\right]$ is unique solution of the equation (40) in $C\left[0, \frac{\pi}{2}\right]$ if and only if

$$\alpha_0 = -\frac{\pi^2 \lambda}{8},\ \ \alpha_1 = 0.$$

2) The function $v(x) = \frac{2}{\pi\lambda},\ x \in \left[0, \frac{\pi}{2}\right]$ is unique solution of the equation (40) in $C\left[0, \frac{\pi}{2}\right]$ if and only if

$$\alpha_0 = -\frac{\pi^2 \lambda}{8} e^{\frac{2}{\pi\lambda}},\ \ \alpha_1 = -\frac{8}{\pi^3 \lambda^3}.$$

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صنف من معادلات فريدهولم التكاملية الخطية و غير الخطية من النوع الثالث

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خلاصة

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