Kuwait J.Sci., Vol.49, No.(3),July.2022,pp(1-16)

# New soliton solutions of nonlinear Kudryashov's equation via Improved $\tan\left(\frac{\phi(\mu)}{2}\right)$-expansion approach in optical fiber

Saima Arshed[1], Nauman Raza[1], Asma Rashid Butt[2], Mustafa Inc[3,4,*]
[1]*Dept. of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore, Pakistan.*
[2]*Dept. of Mathematics, University of Engineering and Technology, Lahore, Pakistan.*
[3] *Dept. of Computer Engineering, Istanbul, Turkey.*
[4*] *Dept. of Medical Research, China Medical University Hospital,*
China Medical University, Taichung, Taiwan.
**Corresponding author: minc@firat.edu.tr*

**Abstract**

In this article, a newly introduced model nonlinear Kudryashov's equation with anti-cubic nonlinearity is considered for extraction of soliton solutions. This model is utilized to depict the propagation of modulated envelope signals which disseminate with some group velocity. To find a solution, an appropriate traveling wave hypothesis is used to covert the given model into a nonlinear ordinary differential equation. An analytical technique, the Improved $\tan\left(\frac{\phi(\mu)}{2}\right)$-expansion approach has been employed on the governing model to construct many new forms of dark soliton, singular soliton, periodic soliton, dark-singular combo soliton and rational solution. The constraint conditions for the existence of these solitons have also been provided. The physical significance of the proposed equation has been provided with a graphical representation of the constructed solutions.

**Keywords:** Anti-cubic nonlinearity; Improved $\tan\left(\frac{\phi(\mu)}{2}\right)$-expansion approach; Kudryashov's equation; Soliton

## 1. Introduction

Nonlinear phenomena are very important in various theories of physical reality. They have applications in solid state material science, liquid mechanics, telecommunications, biology, plasma physical science, oceanography, space science, astronomy, and so forth (W.X. Ma, 2015; W.X. Ma *et al.*, 2016; D. Kumar *et al.*, 2014; J.Y. Yang & W.X. Ma, 2016; A.M. Wazwaz, 2016; K.K. Ali *et al.*, 2020; N. Raza *et al.*, 2018; D. Kumar *et al.*, 2017; A. Javaid *et al.*, 2019; A.M. Wazwaz, 2016; M. Mirzazadeh *et al.*, 2015; S. Arshed & N. Raza, 2020; N. Raza *et al.*, 2019; A.R. Seadawy & D. Lu, 2017; D. Lu *et al.*, 2017), (H.A. Ghany & A.A. Hyder, 2014) and (Harold Exton, 2000). One of the most significant nonlinear phenomenon is the soliton wave. Numerous applications of these waves can be seen in the fields of plasma science, telecommunications and data transmission. A number of models provide a depiction of dynamics of these solitons, including different forms of Korteweg-de Vries as well as Nonlinear Schrodinger equations (M. Mirzazadeh *et al.*, 2015; S. Arshed & N. Raza, 2020; N. Raza *et al.*, 2019; A.R. Seadawy & D. Lu, 2017; D. Lu *et al.*, 2017; K.R. Khusnutdinova *et al.*, 2018; Y. Turbal *et al.*, 2015). These models are utilized to model lone waves with constant velocity which emerge because of the delicate balance of scattering with nonlinearity.

Exact solutions of nonlinear evolution equations are fundamental to understanding the numerous mathematical and physical phenomena. The application of these equations can be seen in almost every branches of science including physics, chemistry and biology at microscopic as well as macroscopic level.

The aim of the current work is to find some new soliton solutions of Kudryashov's equation by using the $tan(\frac{\Phi(\xi)}{2})$ expansion mathod. This method already has been successfully applied to a number of real world problems to understand the phenomenon they exhibit. (J. Manafian & R.F. Zinati, 2020) apply $tan(\frac{\Phi(\xi)}{2})$ expansion mathod to retrieve soliton solutions in the form of hyperbolic function, trigonometric function, and rational function solutions for many nonlinear models including: time fractional Burgers, space-time fractional Fokas, time fractional Cahn-Hilliard, time fractional biological population model, and space-time fractional Whitham-Broer-Kaup equations. (Y. Ugurlu *et al.*, 2017) retrieved traveling wave solutions such as hyperbolic function, exponential function solutions, trigonometric function solutions of the (3 + 1)-dimensional shallow water model and potential KdV equation by using $tan(\frac{\Phi(\xi)}{2})$ expansion method. This technique was also studied by (U. Khan *et al.*, 2018) to investigate (2+1)-dimensional KP-BBM model and reported various kinds of traveling wave solutions. (J. Manafian *et al.*, 2016) investigated (2+1)-dimensional Zoomeron, the symmetric regularized long wave (SRLW) and the Duffing equations by the proposed method.

The improved $\tan(\frac{\Phi(\xi)}{2})$ expansion method is a better choice to study nonlinear evolution equations (NLEEs) as it provides a wide range of solutions to a certain physical model. This methods provides a straight forward procedure to obtain solutions as it does not require initial and boundary conditions. The results obtained in this article are compared with already published literature and demonstrated through remark given at the end of results and discussion section.

The rest of the article is as follows: Section 2 presents the description of the improved $\tan(\frac{\phi(\mu)}{2})$-expansion approach. The governing model has been discussed in section 3. Section 4 presents the soliton solutions of the proposed model. Graphical representations of a few constructed solutions are given in section 5. Conclusions are given in Section 6.

## 2. Description of proposed method

An analytical technique, namely the improved $\tan(\frac{\phi(\mu)}{2})$-expansion approach (N. Raza *et al.*, 2020), has been discussed in detail in the following subsection.

### 2.1 Improved $\tan(\frac{\phi(\mu)}{2})$-expansion approach

Any nonlinear PDE is converted into an ODE by applying the transformation $c(a,b) = C(\mu)$, $\mu = a - hb$, where $h$ is constant.
It is supposed that the converted ODE has a solution of the following form using the improved $\tan(\frac{\phi(\xi)}{2})$-expansion approach

$$C(\mu) = \sum_{r=0}^{m} a_r \left[y + \tan\left(\frac{\phi(\mu)}{2}\right)\right]^r + \sum_{r=1}^{m} b_r \left[y + \tan\left(\frac{\phi(\mu)}{2}\right)\right]^{-r}, \tag{1}$$

where $a_r$ and $b_r$ are constants. $\phi = \phi(\mu)$ satisfies the following ODE

$$\phi'(\mu) = p\sin(\phi(\mu)) + q\cos(\phi(\mu)) + s. \tag{2}$$

The above equation possesses the following solutions
Family 1:
When $\omega = p^2 + q^2 - s^2 < 0$ and $q - s \neq 0$, then

$$\phi(\mu) = 2\ \tan^{-1}\left[\frac{p}{q-s} - \frac{\sqrt{-\omega}}{q-s}\tan\left(\frac{\sqrt{-\omega}}{2}\hat{\mu}\right)\right]. \tag{3}$$

Family 2:
When $\omega = p^2 + q^2 - s^2 > 0$ and $q - s \neq 0$, then

$$\phi(\mu) = 2\ \tan^{-1}\left[\frac{p}{q-s} + \frac{\sqrt{\omega}}{q-s}\tanh\left(\frac{\sqrt{\omega}}{2}\hat{\mu}\right)\right]. \tag{4}$$

Family 3:
When $\Omega = p^2 + q^2 - s^2 > 0$, $q \neq 0$ and $s = 0$, then

$$\phi(\mu) = 2\ \tan^{-1}\left[\frac{p}{q} + \frac{\sqrt{p^2+q^2}}{q}\tanh\left(\frac{\sqrt{p^2+q^2}}{2}\hat{\mu}\right)\right]. \tag{5}$$

Family 4:
When $\Omega = p^2 + q^2 - s^2 < 0$, $s \neq 0$ and $q = 0$, then

$$\phi(\mu) = 2\ \tan^{-1}\left[-\frac{p}{s} + \frac{\sqrt{s^2-p^2}}{s}\tan\left(\frac{\sqrt{s^2-p^2}}{2}\hat{\mu}\right)\right]. \tag{6}$$

Family 5:
When $\Omega = p^2 + q^2 - s^2 > 0$, $q - s \neq 0$ and $p = 0$, then

$$\phi(\mu) = 2\ \tan^{-1}\left[\sqrt{\frac{q+s}{q-s}}\tanh\left(\frac{\sqrt{q^2-s^2}}{2}\hat{\mu}\right)\right]. \tag{7}$$

Family 6:
When $p = 0$ and $s = 0$, then

$$\phi(\mu) = \tan^{-1}\left[\frac{e^{2q\hat{\mu}}-1}{e^{2q\hat{\mu}}+1}, \frac{2e^{q\hat{\mu}}}{e^{2q\hat{\mu}}+1}\right]. \tag{8}$$

Family 7:
When $q = 0$ and $s = 0$, then

$$\phi(\mu) = \tan^{-1}\left[\frac{2e^{p\hat{\mu}}}{e^{2p\hat{\mu}}+1}, \frac{e^{2p\hat{\mu}}-1}{e^{2p\hat{\mu}}+1}\right]. \tag{9}$$

Family 8:
When $p^2 + q^2 = s^2$, then

$$\phi(\mu) = -2\ \tan^{-1}\left[\frac{(q+s)(p\hat{\mu}+2)}{p^2\hat{\mu}}\right]. \tag{10}$$

Family 9:
When $p = q = s = kp$, then

$$\phi(\mu) = 2\ \tan^{-1}\left[e^{kp\hat{\mu}} - 1\right]. \tag{11}$$

Family 10:
When $p = s = kp$ and $q = -kp$, then

$$\phi(\mu) = -2\ \tan^{-1}\left[\frac{e^{kp\hat{\mu}}}{-1+e^{kp\hat{\mu}}}\right]. \tag{12}$$

Family 11:
When $s = p$, then

$$\phi(\mu) = -2\ \tan^{-1}\left[\frac{(p+q)e^{q\hat{\mu}}-1}{(p-q)e^{q\hat{\mu}}-1}\right]. \tag{13}$$

Family 12::
When $p = s$, then

$$\phi(\mu) = 2\ \tan^{-1}\left[\frac{(q+s)e^{q\hat{\mu}}+1}{(q-s)e^{q\hat{\mu}}-1}\right]. \tag{14}$$

Family 13:
When $s = -p$, then

$$\phi(\mu) = 2\ \tan^{-1}\left[\frac{e^{q\hat{\mu}}+q-p}{e^{q\hat{\mu}}-q-p}\right]. \tag{15}$$

Family 14:
When $q = -s$, then

$$\phi(\mu) = -2\ \tan^{-1}\left[\frac{pe^{p\hat{\mu}}}{se^{p\hat{\mu}} - 1}\right]. \tag{16}$$

Family 15:
When $q = 0$ and $p = s$, then

$$\phi(\mu) = -2\ \tan^{-1}\left[\frac{s\hat{\mu} + 2}{s\hat{\mu}}\right]. \tag{17}$$

Family 16:
When $p = 0$ and $q = s$, then

$$\phi(\mu) = 2\ \tan^{-1}[s\hat{\mu}]. \tag{18}$$

Family 17:
When $p = 0$ and $q = -s$, then

$$\phi(\mu) = -2\ \tan^{-1}\left[\frac{1}{s\hat{\mu}}\right]. \tag{19}$$

Family 18:
When $p = 0$ and $q = 0$, then

$$\phi(\mu) = s\hat{\mu}, \tag{20}$$

where $\hat{\mu} = \mu + A$, $a_r(r = 0, 1, 2, \ldots, m)$, $b_r(r = 1, 2, \ldots, m)$, $p$, $q$ and $s$ are unknowns to be determined. The value of $m$ is found using the balancing principle of homogeneity.

Replace Equation (1) with an ODE. Collect the coefficients of same powers of $\tan\left(\frac{\phi(\mu)}{2}\right)$ and $\cot\left(\frac{\phi(\mu)}{2}\right)$ and equate them to zero. A system of algebraic equations is obtained.
Upon solving the system, the unknowns $a_0$, $a_1$, $a_2$, ... , $a_r$, $b_1$, $b_2$, ... , $b_r$ are obtained. The traveling wave solutions are gained by putting the values of unknowns $a_0$, $a_1$, $a_2$, ... , $a_r$, $b_1$, $b_2$, ... , $b_r$ in Equation (1).

## 3. Governing model

In this paper, the governing model which is under consideration for the extraction of soliton solutions is the well known Kudryashov's equation. Kudryashov himself discussed the Kudryashov equation in his article (N.A. Kudryashov, 2019). The dimensionless form of Kudryashov's equation (KE) is

$$ic_b + e_1 c_{aa} + \left(e_2|c|^{-2t} + e_3|c|^{-t} + e_4|c|^t + e_5|c|^{2t}\right)c = 0, \tag{21}$$

here $i = \sqrt{-1}$. In Equation (21), the first term represents temporal evolution, also $e_1,\ e_2,\ e_3,\ e_4$ and $e_5$ are parameters on real line further $t$ represent nonlinear term of arbitrary order. The coefficient $e_1$ is the group velocity dispersion and $e_2,\ e_3,\ e_4$ and $e_5$ are coefficients of nonlinearity that came from the refractive index principle of an optical fiber and give rise to self-phase modulation of the model. If we put $t = 1$ and $e_2 = e_3 = e_4 = 0$, then Equation (21) is converted into a NLSE. For $t = 2$ and $e_4 = 0$, Equation (21) becomes a NLSE with anti-cubic nonlinear term. When $e_2 = e_3 = 0$, the equation (21) is converted into dual-power law of refractive index. For the case $e_2 = e_3 = e_4 = 0$, Equation (21) reduces to the power law of refractive index. In (A. Biswas *et al.*, 2020), bright and singular optical soliton solutions are found for Equation (21).

## 4. Soliton solutions via improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$ expansion method

In the present section, the improved $tan(\frac{\Phi(\xi)}{2})$ expansion method (J. Manafian & R.F. Zinati, 2020) is employed to extract different kinds of explicit solutions including dark soliton, combo soliton, kink soliton, singular soliton solutions. The existence criterion for the newly found solutions is also given.
In order to obtain a traveling wave solution of Kudryashov equation, Equation (21), the following wave transformation is used,

$$c(a, b) = C(\mu)e^{i(wa - kb)}, \qquad \mu = a - vb,$$

where $C(\mu)$ is the shape of the pulse, $wa - kb$ is the phase component and $v$, $w$ and $k$ are the velocity, frequency and wave number of the soliton, respectively.
Plugging the above transformation in Equation (21), to obtain an ODE, as

$$-\iota v C^{2t}C' + kC^{2t}C + e_1 C^{2t}C'' + 2\iota e_1 w C^{2t}C' \qquad -e_1 w^2 C^{2t}C + e_2 C \tag{22}$$
$$+e_3 C^t C + e_4 C^{3t}C + e_5 C^{4t}C = 0. \tag{23}$$

The closed form solution is obtained by using the transformation $C = D^{\frac{1}{t}}$ to Equation (22). Calculate the real and imaginary parts as given below

$$e_1 tDD'' - e_1(t-1)D'^2 + t^2(k - e_1 w^2)D^2 + e_2 t^2 + e_3 t^2 D + e_4 t^2 D^3 + e_5 t^2 D^4 = 0, \tag{24}$$

$$t(2e_1 w - v)DD' = 0, \tag{25}$$

where $D(\mu)$ gives the form of the pulse. Equation (25) provides the speed of the soliton

$$v = 2e_1 w.$$

Balancing $DD''$ and $D^4$ in Equation (24) yields $m = 1$. Using the improved $\tan\left(\frac{\Phi(\xi)}{2}\right)$-expansion approach, the trial solution for $y = 0$ of Equation (24) becomes

$$D(\mu) = a_0 + a_1\left[\tan\left(\frac{\phi(\mu)}{2}\right)\right] + b_1\left[\tan\left(\frac{\phi(\mu)}{2}\right)\right]^{-1}, \tag{26}$$

where $a_0$, $a_1$ and $b_1$ are unknowns. Substituting Equation (26) along with derivatives in Equation (24), a system of algebraic equations is generated by comparing coefficients of alike powers of $\tan\left(\frac{\phi(\mu)}{2}\right)$, equal to zero. After solving the generated system of nonlinear equations, the subsequent solution sets are obtained

**SET 1**

$$a_0 = \frac{-e_4 t(1+t) \pm \sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5 t(2+t)}, \qquad e_1 < 0, e_5 > 0$$
$$a_1 = 0,$$
$$b_1 = \pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}, \qquad e_1 < 0, e_5 > 0$$
$$k = \frac{3e_4^2 t^2(1+t) + e_1 e_5(2+n)^2(p^2+q^2-s^2+2w^2t^2)}{2e_5 t^2(2+t^2)}$$

**SET 2**

$$a_0 = \frac{-e_4 t(1+t) \pm \sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5 t(2+t)}, \qquad e_1 < 0, e_5 > 0$$
$$a_1 = \mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}, \qquad e_1 < 0, e_5 > 0$$
$$b_1 = 0,$$
$$k = \frac{3e_4^2 t^2(1+t) + e_1 e_5(2+n)^2(p^2+q^2-s^2+2w^2t^2)}{2e_5 t^2(2+t^2)}$$

**SET 3**

$$a_0 = \frac{-e_4 t(1+t) \pm \sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5 t(2+t)}, \qquad e_1 < 0, e_5 > 0$$

$$a_1 = \mp \frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}, \qquad e_1 < 0, e_5 > 0$$
$$b_1 = \pm \frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}, \qquad e_1 < 0, e_5 > 0$$
$$k = \frac{3e_4^2t^2(1+t) + e_1e_5(2+n)^2(p^2 + 2(-q^2 + s^2 + w^2t^2))}{2e_5t^2(2+t^2)}$$

where $p$, $q$ and $s$ are arbitrary constants. Using Equation (26) and **SET 1**, the family of solutions 1, 2, 3, 4 and 5 are given as

$$\begin{aligned} c_{1,1}(a,b) &= e^{i(wa-kb)}\left[\frac{-e_4t(1+t) \pm \sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right. \\ &\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q-s} - \frac{\sqrt{-\omega}}{q-s}\tan\left(\frac{\sqrt{-\omega}}{2}\hat{\mu}\right)\right]^{-1}\right]^{\frac{1}{t}}, \\ c_{1,2}(a,b) &= e^{i(wa-kb)}\left[\frac{-e_4t(1+t) \pm \sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right. \\ &\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q-s} + \frac{\sqrt{\omega}}{q-s}\tanh\left(\frac{\sqrt{\omega}}{2}\hat{\mu}\right)\right]^{-1}\right]^{\frac{1}{t}}, \\ c_{1,3}(a,b) &= e^{i(wa-kb)}\left[\frac{-e_4t(1+t) \pm \sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right. \\ &\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q} + \frac{\sqrt{p^2+q^2}}{q}\tanh\left(\frac{\sqrt{p^2+q^2}}{2}\hat{\mu}\right)\right]^{-1}\right]^{\frac{1}{t}}, \\ c_{1,4}(a,b) &= e^{i(wa-kb)}\left[\frac{-e_4t(1+t) \pm \sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right. \\ &\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[-\frac{p}{s} + \frac{\sqrt{s^2-p^2}}{s}\tan\left(\frac{\sqrt{s^2-p^2}}{2}\hat{\mu}\right)\right]^{-1}\right]^{\frac{1}{t}}, \\ c_{1,5}(a,b) &= e^{i(wa-kb)}\left[\frac{-e_4t(1+t) \pm \sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right. \\ &\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\sqrt{\frac{q+s}{q-s}}\tanh\left(\frac{\sqrt{q^2-s^2}}{2}\hat{\mu}\right)\right]^{-1}\right]^{\frac{1}{t}}, \end{aligned}$$

here $\hat{\xi} = \xi + C$.
Using Equation (26) and **SET 1**, the solution family 6, 7, 8, 9 and 10 is obtained as

$$c_{1,6}(a,b) = e^{i(wa-kb)}\left[\frac{-e_4t(1+t) \pm \sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$
$$\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\tan\left\{\frac{1}{2}\arctan\left(\frac{e^{2q\hat{\mu}}-1}{e^{2q\hat{\mu}}+1}, \frac{2e^{q\hat{\mu}}}{e^{2q\hat{\mu}}+1}\right)\right\}\right]^{-1}\right]^{\frac{1}{t}},$$
$$c_{1,7}(a,b) = e^{i(wa-kb)}\left[\frac{-e_4t(1+t) \pm \sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$
$$\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\tan\left\{\frac{1}{2}\arctan\left(\frac{2e^{p\hat{\mu}}}{e^{2p\hat{\mu}}+1}, \frac{e^{2p\hat{\mu}}-1}{e^{2p\hat{\mu}}+1}\right)\right\}\right]^{-1}\right]^{\frac{1}{t}},$$
$$c_{1,8}(a,b) = e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(q+s)(p\hat{\mu}+2)}{p^2\hat{\mu}}\right]^{-1}\right]^{\frac{1}{t}},$$

$$c_{1,9}(a,b)=e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}[e^{kp\hat{\mu}}-1]^{-1}\right]^{\frac{1}{t}},$$

$$c_{1,10}(a,b)=e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{e^{kp\hat{\mu}}}{-1+e^{kp\hat{\mu}}}\right]^{-1}\right]^{\frac{1}{t}},$$

where $\hat{\mu}=\mu+A$.
For Equation (26) and **set 1**, the solution family 11, 12, 13, 14, 15, 16 and 17 is obtained as

$$c_{1,11}(a,b)=e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(p+q)e^{q\hat{\mu}}-1}{(p-q)e^{q\hat{\mu}}-1}\right]^{-1}\right]^{\frac{1}{t}},$$

$$c_{1,12}(a,b)=e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(q+s)e^{q\hat{\mu}}+1}{(q-s)e^{q\hat{\mu}}-1}\right]^{-1}\right]^{\frac{1}{t}},$$

$$c_{1,13}(a,b)=e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{e^{q\hat{\xi}}+q-p}{e^{g\hat{\mu}}-q-p}\right]^{-1}\right]^{\frac{1}{t}},$$

$$c_{1,14}(a,b)=e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{pe^{p\hat{\mu}}}{se^{p\hat{\mu}}-1}\right]^{-1}\right]^{\frac{1}{t}},$$

$$c_{1,15}(a,b)=e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{s\hat{\mu}+2}{s\hat{\mu}}\right]^{-1}\right]^{\frac{1}{t}},$$

$$c_{1,16}(a,b)=e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[s\hat{\mu}\right]^{-1}\right]^{\frac{1}{t}},$$

$$c_{1,17}(a,b)=e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{1}{s\hat{\mu}}\right]^{-1}\right]^{\frac{1}{t}},$$

where $\hat{\mu}=\mu+A$.
Using Equation (**??**) and **SET 2**, the solution family 1, 2, 3, 4 and 5 is obtained as

$$c_{2,1}(a,b) \quad = \quad e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$

$$\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q-s}-\frac{\sqrt{-\omega}}{q-s}\tan\left(\frac{\sqrt{-\omega}}{2}\hat{\mu}\right)\right]\Bigg]^{\frac{1}{t}},$$

$$c_{2,2}(a,b) = e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$
$$\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q-s}+\frac{\sqrt{\omega}}{q-s}\tanh\left(\frac{\sqrt{\omega}}{2}\hat{\mu}\right)\right]\Bigg]^{\frac{1}{t}},$$

$$c_{2,3}(a,b) = e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$
$$\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q}+\frac{\sqrt{p^2+q^2}}{q}\tanh\left(\frac{\sqrt{p^2+q^2}}{2}\hat{\mu}\right)\right]\Bigg]^{\frac{1}{t}},$$

$$c_{2,4}(a,b) = e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$
$$\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[-\frac{p}{s}+\frac{\sqrt{s^2-p^2}}{s}\tan\left(\frac{\sqrt{s^2-p^2}}{2}\hat{\mu}\right)\right]\Bigg]^{\frac{1}{t}},$$

$$c_{2,5}(a,b) = e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$
$$\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\sqrt{\frac{q+s}{q-s}}\tanh\left(\frac{\sqrt{q^2-s^2}}{2}\hat{\mu}\right)\right]\Bigg]^{\frac{1}{t}},$$

where $\hat{\mu}=\mu+C$.
Using Equation (26) and **SET 2**, the solution family 6, 7, 8, 9 and 10 is obtained as

$$c_{2,6}(a,b) = e^{i(wa-kb)}$$
$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$
$$\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\tan\left\{\frac{1}{2}\arctan\left(\frac{e^{2q\hat{\mu}}-1}{e^{2q\hat{\mu}}+1},\frac{2e^{q\hat{\mu}}}{e^{2q\hat{\mu}}+1}\right)\right\}\right]\Bigg]^{\frac{1}{t}},$$

$$c_{2,7}(a,b) = e^{i(wa-kb)}$$
$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$
$$\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\tan\left\{\frac{1}{2}\arctan\left(\frac{2e^{p\hat{\mu}}}{e^{2p\hat{\mu}}+1},\frac{e^{2p\hat{\mu}}-1}{e^{2p\hat{\mu}}+1}\right)\right\}\right]\Bigg]^{\frac{1}{t}},$$

$$c_{2,8}(a,b) = e^{i(wa-kb)}$$
$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(q+s)(p\hat{\mu}+2)}{p^2\hat{\mu}}\right]\right]^{\frac{1}{t}},$$

$$c_{2,9}(a,b) = e^{i(wa-kb)}$$
$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}[e^{kp\hat{\mu}}-1]\right]^{\frac{1}{t}},$$

$$c_{2,10}(a,b) = e^{i(wa-kb)}$$

$$\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{e^{kp\hat{\mu}}}{-1+e^{kp\hat{\mu}}}\right]\right]^{\frac{1}{t}},$$

where $\hat{\mu}=\mu+A$.
For Equation (26) and **set 2**, the solution family 11, 12, 13, 14, 15, 16 and 17 given as

$$c_{2,11}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$
$$\left.\pm\frac{(q-s)\sqrt{-e_1(1+t)}}{2t\sqrt{e_5}}\left[\frac{(p+q)e^{q\hat{\mu}}-1}{(p-q)e^{q\hat{\mu}}-1}\right]^{\frac{1}{t}}\right.,$$

$$c_{2,12}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(q+s)e^{q\hat{\mu}}+1}{(q-s)e^{q\hat{\mu}}-1}\right]\right]^{\frac{1}{t}},$$
$$c_{2,13}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{e^{q\hat{\xi}}+q-p}{e^{g\hat{\mu}}-q-p}\right]\right]^{\frac{1}{t}},$$

$$c_{2,14}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.$$
$$\left.\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{pe^{p\hat{\mu}}}{se^{p\hat{\mu}}-1}\right]^{\frac{1}{t}}\right.,$$

$$c_{2,15}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{s\hat{\mu}+2}{s\hat{\mu}}\right]\right]^{\frac{1}{t}},$$
$$c_{2,16}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[s\hat{\mu}\right]\right]^{\frac{1}{t}},$$
$$c_{2,17}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{1}{s\hat{\mu}}\right]\right]^{\frac{1}{t}},$$

where $\hat{\mu}=\mu+A$.
Using Equation (26) and **SET 3**, the solution family 1, 2, 3, 4 and 5 is obtained as

$$\begin{aligned}c_{3,1}(a,b) &= e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.\\ &\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q-s}-\frac{\sqrt{-\omega}}{q-s}\tan\left(\frac{\sqrt{-\omega}}{2}\hat{\mu}\right)\right]\\ &\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q-s}-\frac{\sqrt{-\omega}}{q-s}\tan\left(\frac{\sqrt{-\omega}}{2}\hat{\mu}\right)\right]^{-1}\right]^{\frac{1}{t}},\\ c_{3,2}(a,b) &= e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\right.\\ &\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q-s}+\frac{\sqrt{\omega}}{q-s}\tanh\left(\frac{\sqrt{\omega}}{2}\hat{\mu}\right)\right]\end{aligned}$$

$$\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q-s}+\frac{\sqrt{\omega}}{q-s}\tanh\left(\frac{\sqrt{\omega}}{2}\hat{\mu}\right)\right]^{-1}\Bigg]^{\frac{1}{t}},$$

$$\begin{aligned}
c_{3,3}(a,b) &= e^{i(wa-kb)}\Bigg[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\\
&\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q}+\frac{\sqrt{p^2+q^2}}{q}\tanh\left(\frac{\sqrt{p^2+q^2}}{2}\hat{\mu}\right)\right]\\
&\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{p}{q}+\frac{\sqrt{p^2+q^2}}{q}\tanh\left(\frac{\sqrt{p^2+q^2}}{2}\hat{\mu}\right)\right]^{-1}\Bigg]^{\frac{1}{t}},\\
c_{3,4}(a,b) &= e^{i(wa-kb)}\Bigg[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\\
&\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[-\frac{p}{s}+\frac{\sqrt{s^2-p^2}}{s}\tan\left(\frac{\sqrt{s^2-p^2}}{2}\hat{\mu}\right)\right]\\
&\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[-\frac{p}{s}+\frac{\sqrt{s^2-p^2}}{s}\tan\left(\frac{\sqrt{s^2-p^2}}{2}\hat{\mu}\right)\right]^{-1}\Bigg]^{\frac{1}{t}},\\
c_{3,5}(a,b) &= e^{i(wa-kb)}\Bigg[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\\
&\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\sqrt{\frac{q+s}{q-s}}\tanh\left(\frac{\sqrt{q^2-s^2}}{2}\hat{\mu}\right)\right]\\
&\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\sqrt{\frac{q+s}{q-s}}\tanh\left(\frac{\sqrt{q^2-s^2}}{2}\hat{\mu}\right)\right]^{-1}\Bigg]^{\frac{1}{t}},
\end{aligned}$$

where $\hat{\mu}=\mu+C$.
Using Equation (26) and **SET 3**, the solution family 6, 7, 8, 9 and 10 is obtained as

$$c_{3,6}(a,b)=e^{i(wa-kb)}\Bigg[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}$$

$$\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\tan\left\{\frac{1}{2}\arctan\left(\frac{e^{2q\hat{\mu}}-1}{e^{2q\hat{\mu}}+1},\frac{2e^{q\hat{\mu}}}{e^{2q\hat{\mu}}+1}\right)\right\}\right]$$

$$\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\tan\left\{\frac{1}{2}\arctan\left(\frac{e^{2q\hat{\mu}}-1}{e^{2q\hat{\mu}}+1},\frac{2e^{q\hat{\mu}}}{e^{2q\hat{\mu}}+1}\right)\right\}\right]^{-1}\Bigg]^{\frac{1}{t}},$$

$$c_{3,7}(a,b)=e^{i(wa-kb)}\Bigg[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}$$

$$\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\tan\left\{\frac{1}{2}\arctan\left(\frac{2e^{p\hat{\mu}}}{e^{2p\hat{\mu}}+1},\frac{e^{2p\hat{\mu}}-1}{e^{2p\hat{\mu}}+1}\right)\right\}\right]$$

$$\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\tan\left\{\frac{1}{2}\arctan\left(\frac{2e^{p\hat{\mu}}}{e^{2p\hat{\mu}}+1},\frac{e^{2p\hat{\mu}}-1}{e^{2p\hat{\mu}}+1}\right)\right\}\right]^{-1}\Bigg]^{\frac{1}{t}},$$

$$c_{3,8}(a,b)=e^{i(wa-kb)}$$

$$\Bigg[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(q+s)(p\hat{\mu}+2)}{p^2\hat{\mu}}\right]$$

$$\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(q+s)(p\hat{\mu}+2)}{p^2\hat{\mu}}\right]^{-1}\Bigg]^{\frac{1}{t}},$$

$$c_{3,9}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}[e^{kp\hat{\mu}}-1]\right.$$
$$\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}[e^{kp\hat{\mu}}-1]^{-1}\right]^{\frac{1}{t}},$$
$$c_{3,10}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{e^{kp\hat{\mu}}}{-1+e^{kp\hat{\mu}}}\right]\right.$$
$$\left.\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{e^{kp\hat{\mu}}}{-1+e^{kp\hat{\mu}}}\right]^{-1}\right]^{\frac{1}{t}},$$

where $\hat{\mu}=\mu+A$.
For Equation (26) and **set 3**, the solution family 11, 12, 13, 14, 15, 16 and 17 given as

$$c_{3,11}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(p+q)e^{q\hat{\mu}}-1}{(p-q)e^{q\hat{\mu}}-1}\right]\right.$$
$$\left.\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(p+q)e^{q\hat{\mu}}-1}{(p-q)e^{q\hat{\mu}}-1}\right]^{-1}\right]^{\frac{1}{t}},$$
$$c_{3,12}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(q+s)e^{q\hat{\mu}}+1}{(q-s)e^{q\hat{\mu}}-1}\right]\right.$$
$$\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{(q+s)e^{q\hat{\mu}}+1}{(q-s)e^{q\hat{\mu}}-1}\right]^{-1}\right]^{\frac{1}{t}},$$
$$c_{3,13}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{e^{q\hat{\xi}}+q-p}{e^{g\hat{\mu}}-q-p}\right]\right.$$
$$\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{e^{q\hat{\xi}}+q-p}{e^{g\hat{\mu}}-q-p}\right]^{-1}\right]^{\frac{1}{t}},$$

$$c_{3,14}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{pe^{p\hat{\mu}}}{se^{p\hat{\mu}}-1}\right]\right.$$
$$\left.\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{pe^{p\hat{\mu}}}{se^{p\hat{\mu}}-1}\right]^{-1}\right]^{\frac{1}{t}},$$
$$c_{3,15}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{s\hat{\mu}+2}{s\hat{\mu}}\right]\right.$$
$$\left.\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{s\hat{\mu}+2}{s\hat{\mu}}\right]^{-1}\right]^{\frac{1}{t}},$$
$$c_{3,16}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\mp\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[s\hat{\mu}\right]\right.$$
$$\left.\pm\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[s\hat{\mu}\right]^{-1}\right]^{\frac{1}{t}},$$
$$c_{3,17}(a,b)=e^{i(wa-kb)}\left[\frac{-e_4t(1+t)\pm\sqrt{-e_1}\sqrt{e_5(1+t)}p(2+t)}{2e_5t(2+t)}\pm\frac{\sqrt{-e_1}(q-s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{1}{s\hat{\mu}}\right]\right.$$
$$\left.\mp\frac{\sqrt{-e_1}(q+s)\sqrt{1+t}}{2t\sqrt{e_5}}\left[\frac{1}{s\hat{\mu}}\right]^{-1}\right]^{\frac{1}{t}},$$

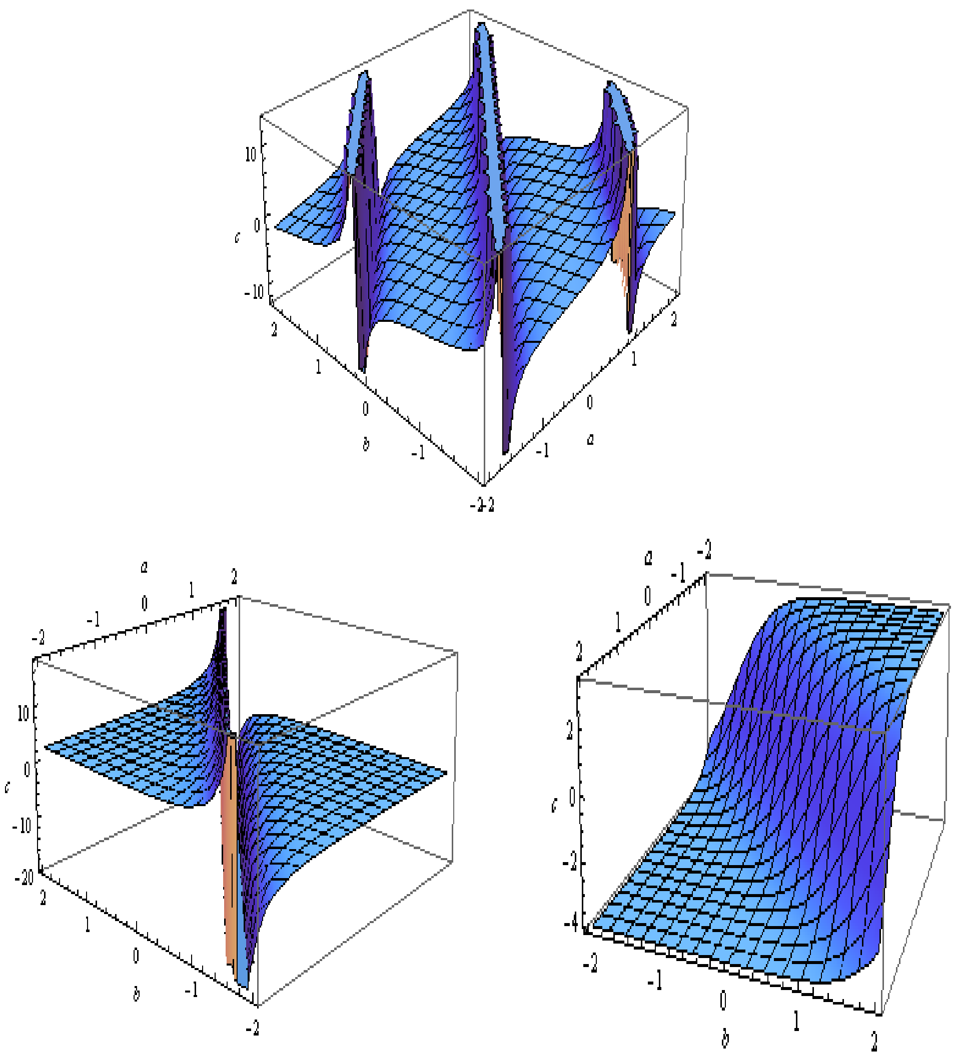

**Fig. 1.** 3D plots of solutions for **SET 1** for $|c_{1,1}|$, $|c_{1,3}|$ and $|c_{1,12}|$ respectively.

where $\hat{\mu} = \mu + A$.
All the above constructed solutions are valid for $e_1 < 0$ and $e_5 > 0$.

## 5. Graphical depiction of the obtained results

This section contains the 3D graphs of a few constructed soliton solutions corresponding to **SET 1**, **SET 2** and **SET 3**. The solutions in 3D graphs depict the physical profile of propagating wave. Dark soliton, singular soliton, combo soliton and periodic soliton are obtained on choosing particular values to arbitrary parameters. The graphs presented in this section, are very helpful in visualizing the underlying mechanism of the proposed model. Using mathematical software Mathematica, three dimensional plots of some obtained exact traveling wave solutions have been shown in Figures 1- 3. Figure 1 presents periodic, singular and kink soliton solutions for $e_1 = -1, e_4 = 1, t = 1, q = 3, p = 1, e_5 = 1$ and $s = 4, s = 0, s = 1$, respectively. Figure 2 presents periodic and dark soliton solutions for $e_1 = -1, e_4 = 1, t = 0.5, q = 3, s = 4, p = 1, e_5 = 1$, and $s = 4, s = 0, s = 1$, respectively. Figure 3, presents periodic soliton and dark-singular combo soliton solutions for $e_1 = -1, e_4 = 1, t = 0.5, q = 3, p = 1, e_5 = 1$ and $s = 0, s = 4$.

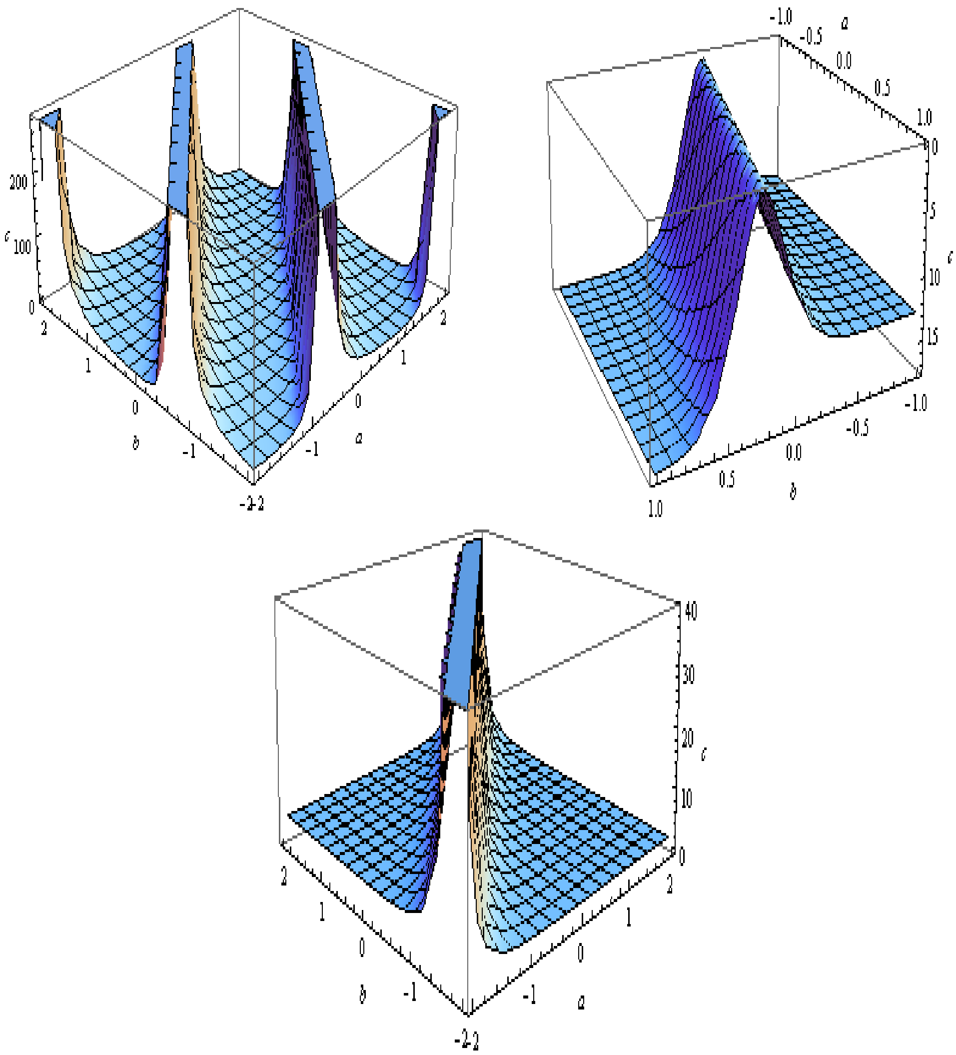

**Fig. 2.** 3D plots of solutions for **SET 2** for $|c_{2,1}|$, $|c_{2,3}|$ and $|c_{2,12}|$ respectively.

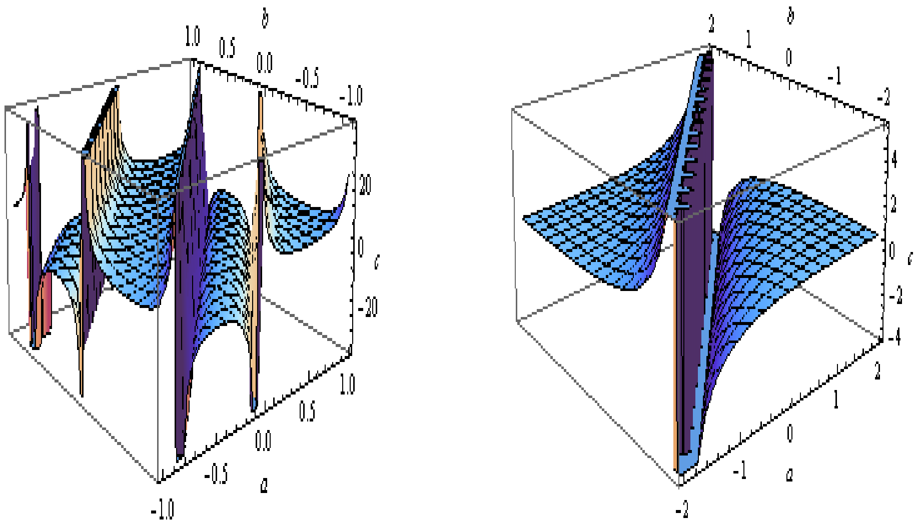

**Fig. 3.** 3D plots of solutions for **SET 3** for $|c_{3,1}|$ and $|c_{3,3}|$ respectively.

5.1 Results and discussion

In this article, the extraction of solitary waves is obtained by the improved $\tan\left(\frac{\phi(\mu)}{2}\right)$-expansion approach. The nonlinear Kudryashov's equation is examined with nonlinearity parameter $t$ using the proposed analytical method. The improved $\tan\left(\frac{\phi(\mu)}{2}\right)$-expansion approach yields dark solitons, singular solitons, dark-singular combo solitons and periodic solitons. Three dimensional plots of few constructed soliton solutions are presented in Figs. 1-3 corresponding to **SET 1**- **SET 3** respectively, by choosing particular values to the unknown parameters. The proposed method provides new explicit form of solitary wave solutions of Kudryashov's equation that have not reported before. The closed-form wave solutions play a significant role in studying the complex physical phenomena found in applied mathematics, mathematical physics, engineering and science.

**Remark 1.** It is important to mentioned here that all the solutions constructed in this paper are more recent than those in (A. Biswas *et al.*, 2020), (N.A. Kudryashov, 2019), (A.H. Arnous *et al.*, 2021; A. Biswas *et al.*, 2020; E.M.E. Zayed *et al.*, 2020; A. Biswas *et al.*, 2020). To the best of our knowledge, the obtained soliton solutions found in this paper are being presented for the first time.

## 6. Conclusion

This investigation obtained some new explicit, localized and periodic traveling wave solutions in the context of the Kudryshov equation under the effect of anti-cubic nonlinearity. This model is a generalization of the anti-cubic law nonlinearity arising in fiber optics communication due to change in refractive index. The proposed model is reduced to an ODE by utilizing the wave transformation to describe the propagation of nonlinear waves that can pass and disseminate from dispersive and nonlinear media (plasma physics, optical fiber etc.). The governing model has been solved with the help of the improved $\tan\left(\frac{\phi(\mu)}{2}\right)$-expansion to reveal many new forms of dark, periodic, combo and singular soliton solutions with their existence criterion. The graphs show the physical significance of the proposed model. It is worth mentioning with the help of Maple software, we have guaranteed the results by putting them back into the original equation. The new type of exact solutions obtained in this letter might have significant impact on future researches.

## References

**A. Javaid, N. Raza, M.S. Osman (2019)**. Multi-solitons of thermophoretic motion equation depicting the wrinkle propagation in substrate-supported graphene sheets. *Communication in Theoretical Physics*, 71: 362-366.

**A. Biswas, M. Ekici, A. Sonmezoglu, A.S. Alshomrani, M.R. Belic (2020)**. Optical solitons with Kudryashov's equation by extended trial function. *Optik*, 202: 163290.

**A. Biswas, M. Asma, P. Guggilla, L. Mullick, L. Moraru, M.Ekici, A.K. Alzahrani, M.R.Belic (2020)**. Optical soliton perturbation with Kudryashov's equation by semi-inverse variational principle. *Physics Letters A*, 384 (33): 126830.

**A. Biswas, J.V. Guzman, M. Ekici, Q. Zhou, H. Triki, A.S. Alshomrani, M.R. Belic (2020)**. Optical solitons and conservation laws of Kudryashov's equation using undetermined coefficients. *Optik*, 202: 163417.

**A.H. Arnous, A. Biswas, M. Ekici, A.K. Alzahrani, M.R. Belic (2021)**. Optical solitons and conservation laws of Kudryashov's equation with improved modified extended tanh-function. *Optik*, 225: 165406.

**A.M. Wazwaz (2016)**. Gaussian solitary wave solutions for nonlinear evolution equations with logarithmic nonlinearities. *Nonlinear Dynamics*, 83: 591-596.

**A.M. Wazwaz (2016)**. Multiple soliton solutions and multiple complex soliton solutions for two distinct Boussinesq equations. *Nonlinear Dynamics*, 85: (2016), 731-737.

**A.R. Seadawy, D. Lu (2017)**. Bright and dark solitary wave soliton solutions for the generalized higher order nonlinear Schrodinger equation and its stability. *Results in Physics*, 7: 43-48.

**D. Kumar, J. Singh, S. Kumar, Sushila. (2014)**. Numerical computation of Klein-Gordon equations arising in quantum field theory by using homotopy analysis transform method. *Alexandria Engineering Journal*, 53: 469-474.

**D. Kumar, K. Hosseini, F. Samadani (2017)**. The sine-Gordon expansion method to look for the traveling wave solutions of the Tzitzeica type equations in nonlinear optics. *Optik*, 149: 439-446.

**D. Lu, A.R. Seadawy, M. Arshad (2017)**. Applications of extended simple equation method on unstable nonlinear Schrodinger equations. *Optik*, 140: 136-144.

**E.M.E. Zayed, R.M.A. Shohib, A. Biswas, M. Ekici, L. Moraru, A.K. Alzahrani, M.R. Belic (2020)**. Optical solitons with differential group delay for Kudryashov's model by the auxiliary equation mapping method. *Chinese Journal of Physics*, 67: 631-645.

**H.A. Ghany** & **A.A. Hyder (2014)**. Exact solutions for the wick-type stochastic time-fractional KdV equations. *Kuwait Journal of Science*, 41(1): 75-84.

**H. Exton (2000)**. On the solution of two mutually coupled Bessel-Clifford equations assosicated with vibrations constrained by cylindrical boundaries. *Kuwait Journal of Science and Engineering*, 27(1): 1-6.

**J. Manafian, R.F. Zinati (2020)**. Application of $\tan\left(\frac{\phi(\mu)}{2}\right)$-expansion method to solve some nonlinear fractional physical model. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, 90: 67-86.

**J. Manafian, M. Lakestani, A. Bekir (2016)**. Study of the analytical treatment of the (2+1)-dimensional Zoomeron, the Duffing and the SRLW equations via a new analytical approach. *International Journal of Applied and Computational Mathematics*, 2: 243.

**J.Y. Yang, W.X. Ma (2016)**. Lump solutions to the BKP equation bysymbolic computation. *International Journal of Modern Physics B*, 30: 28-29.

**K.K. Ali, A.M. Wazwaz, M.S. Osman (2020)**. Optical soliton solutions to the generalized nonautonomous nonlinear Schrödinger equations in optical fibers via the sine-Gordon expansion method. *Optik*, 208: 164132.

**K.R. Khusnutdinova, Y.A. Stepanyants, M.R. Tranter (2018)**. Soliton solutions to the fifth-order Korteweg-de Vries equation and their applications to surface and internal water waves. *Physics of Fluids*, 30: 022104.

**M. Mirzazadeh, A.H. Arnous, M.F. Mahmood, E. Zerrad, A. Biswas (2015)**. Soliton solutions to resonant nonlinear Schrödinger's equation with time-dependent coefficients by trial solution approach. *Nonlinear Dynamics*, 81: 277-282.

**N. Raza, S. Sial, M. Kaplan (2018)**. Exact periodic and explicit solutions of higher dimensional equations with fractional temporal evolution. *Optik*, 156: 628-634.

**N. Raza, S. Arshed, S. Sial (2019)**. Optical solitons for coupled Fokas-Lenells equation in birefringence fibers. *Modern Physics Letters B*, 33: 1950317.

**N. Raza, J. Afzal, A. Bekir, H. Rezazadeh (2020)**. Improved $\tan(\frac{\Phi(\xi)}{2})$-expansion approach for Burgers equation in nonlinear dynamical model of ion acoustic waves. *Brazilian Journal of Physics*, 50: 254-262.

**N.A. Kudryashov (2019)**. A generalized model for description of propagation pulses in optical fiber. *Optik*, 189: 42-52.

**S. Arshed, N. Raza (2020)**. Optical solitons perturbation of Fokas-Lenells equation with full nonlinearity and dual dispersion. *Chinese Journal of Physics*, 63: 314-324.

**U. Khan, A. Irshad, N. Ahmed, S.T. Mohyud-Din (2018)**. Improved $\tan(\frac{\phi(\mu)}{2})$-expansion method for (2 + 1)-dimensional KPBBM wave equation. *Optical Quantum Electronics*, 50: 135.

**W.X. Ma. (2015)**. Lump solutions to the Kadomtsev-Petviashvili equation. *Physics Letters A*, 379: 1975-1978.

**W.X. Ma, Z. Qin, X. Li. (2016)**. Lump solutions to dimensionally reduced p-gKP and p-gBKP equations. *Nonlinear Dynamics*, 84: 923-931.

**Y. Turbal, A. Bomba, M. Turbal (2015)**. Method for Studying the Multisoliton Solutions of the Korteweg-de Vries Type equations. *Journal of Difference Equations*, 2015: 703039.

**Y. Ugurlu, I.E. Inan, H. Bulut (2017)**. Two new applications of $\tan(\frac{F(\mu)}{2})$-expansion method. *Optik* , 131: 539-546.

**Submitted:** 13/02/2021
**Revised:** 08/08/2021
**Accepted:** 26/07/2021
**DOI:** 10.48129/kjs.12441