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Generalized constant ratio surfaces and quaternions

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Absract

In this study, we show that the generalized constant ratio surfaces can be obtained by the quaternion product of a unit quaternion and a pure quaternion. Also, the quaternion product of the unit quaternion and the position vector of the generalized constant ratio surface as a pure quaternion is a generalized constant ratio surface. Then we give some results about the generalized constant ratio surfaces by using the quaternions.

Keywords: Generalized constant ratio surfaces; kinematic; rotation matrices; real quaternions; unit quaternions.

Mathematics Subject Classication (2010): 53A05, 53C45, 53A05, 53A17.

1. Introduction

Hamilton interpreted the complex numbers as points in a plane. Then he described quaternions as the extention to complex numbers (Hamilton, 1844). Hamilton defined quaternions under the following multiplication rules:

$$\begin{aligned} &i^2 = j^2 = k^2 = -1,\ ij = k,\ ji = -k, \\ &jk = i, kj = -i,\ ki = j,\ ik = -j. \end{aligned} \tag{1}$$

Computing rotations by using three Euler angles is not ideal in geometry. Shoemake described the system of rotation by using quaternions known as four coordinate system (Shoemake, 1985). These four coordinate system is more usable for rotations about an arbitrary axis than rotation matrices. By using quaternions, many laws in classical and mechanical physics can be easily given. Quaternions are especially used in three-dimensional rotations such as in computer graphics and computer visions.

Kinematic studies the trajectory of points, lines and other geometric objects and their differential properties. When a point moves just related to one parameter, it traces on 1-dimensional path as its curve. Similarly, Düldül show that when segment of line or rectangle moves on one parameter, they sweep 2-dimensional and 3-dimensional paths, respectively (Düldül, 2010). Since the 2 and 3-dimensional paths are related to properties of curve, the position vector of curve, which describe the characteristic of the curve, is important.

The constant angle surfaces for which their normal make a constant angle with some fixed direction were intensively studied by mathematicans in differential geometry, in recent years. There are a lot of results obtained so far. Dillen *et al*. classified constant angle surfaces in the product manifold $\mathbb{S}^2 \times \mathbb{R}$ (Dillen *et al*., 2007). Dillen & Munteanu classified constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$ (Dillen & Munteanu, 2009). Munteanu & Nistor researched the these surfaces in $\mathbb{E}^3$ (Munteanu & Nistor, 2009).

The position vector has very important role for which it describes properties of surfaces in the Euclidean 3-space. The surfaces in Euclidean space $\mathbb{R}^3$, which have constant angle between the normal and the position vector, were classied by Munteanu (2010). These surfaces were called constant slope surfaces. The tangential component of the position vector onto the tangent plane of the constant slope surfaces is a principal direction. Fu & Munteanu recently generalized the concept of constant slope surfaces (Fu & Munteanu, 2014). These surfaces were called generalized constant ratio (GCR) surfaces. The angle between the normal and the position vector of GCR surfaces is no longer constant. The tangential component of the position vector on the tangent plane of GCR surfaces is the principal direction.

Babaarslan & Yayli showed that constant slope surfaces can be obtained by the quaternion product and by the matrix representations (Babaarslan & Yayli , 2012). In this study, we show that the rotation of the GCR surfaces by the unit quaternion is a GCR surface. Also, these surfaces can be obtained by quaternion product and by the matrix representations. Finally, it is given some results and examples about the GCR sufaces by using quaternion product and the matrix representations.

2. Preliminaries

The set $\mathbb{H}=\{q=a_0 1+a_1 i+a_2 j+a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ of real quaternions is equal to four-dimensional vector space $\mathbb{R}^4$. Quaternions have a basis $\{1,i,j,k\}$ shortly given with some properties as

$$i^2=j^2=k^2=ijk=-1. \tag{2}$$

Thus, the set of real quaternions is associative and not commutative algebra. 1 is identity element of $\mathbb{H}$. As the scalar and the vectorial component of q are $a_0=S_q$ and $w=V_q$, respectively. So we can write the quaternion as $q=S_q+V_q$ and its conjugate as $\overline{q}=S_q-V_q$. If a quaternion has a zero scalar component, it is called pure quaternion. Summation of two quaternions and multiplication of a quaternion with a scalar can be expressed, respectively as

$$\begin{aligned} q&=S_q+V_q,\ p=S_p+V_p\in\mathbb{H} \text{ and } \lambda\in\mathbb{R},\\ q+p&=(S_q+S_p)+(V_q+V_p),\\ \lambda q&=\lambda S_q+\lambda V_q. \end{aligned}$$

Using dot and cross-product in $\mathbb{R}^3$, the quaternion product of two quaternions $q=S_q+V_q$ and $p=S_p+V_p$ is given by

$$q\times p=S_qS_p-\langle V_q,V_p\rangle+S_qV_p+S_pV_q+V_q\wedge V_p, \tag{3}$$

where $\times$ quaternion product. For any two quaternions q and p, the quaternion product have some features defined as

$$\begin{aligned} \overline{q+p}&=\overline{q}+\overline{p} \text{ and } \overline{q\times p}=\overline{p}\times\overline{q},\\ \|q\|&=\overline{q}\times q=q\times\overline{q}=a_0^2+a_1^2+a_2^2+a_3^2. \end{aligned}$$

If $\|q\|=1$, q is called unit quaternion. The inverse of the quaternion q can be given as

$$q^{-1}=\frac{\overline{q}}{\|q\|},\ \|q\|\neq 0.$$

If $\|q\|=1$, q also can be written in the trigonometric form as $q=\cos\theta+\sin\theta v$, where $v\in\mathbb{R}^3$ and $\|v\|=1$. Let $q=\cos\theta+\sin\theta v$ and $p=\cos\varphi+\sin\varphi v$ be two unit quaternions. So, $q\times p=\cos(\theta+\varphi)+\sin(\theta+\varphi)v$ and the power of q is $q^n=\cos(n\theta)+\sin(n\theta)v$, where $n\in\mathbb{R}$ (Bekar & Yayli, 2013).

The 1-parameter homothetic motion in the Euclidean 3-space can be given as

$$y=hAx+C, \tag{4}$$

where A is an orthogonal 3×3-matrix, C is a translation vector, and h is a homothetic scalar. y and x are the position vectors of a same point of the fixed space $\mathbb{R}'$ and the moving space $\mathbb{R}$, respectively. h, A and $C$are continuously differentiable functions of a same real parameters (Düldül, 2010).

Rotations in the three-dimensional space are the most important contribution of quaternions to kinematics, computer graphics and character animations. Let $q\in\mathbb{H}$ be a unit quaternion and v be a pure quaternion (3-dimensional vector in $\mathbb{E}^3$). Then, we can define linear mapping φ as

$$\varphi:\mathbb{E}^3\longrightarrow\mathbb{E}^3,\ \varphi(v)=q\times v\times q^{-1}. \tag{5}$$

Let $q=a_0 1+a_1 i+a_2 j+a_3 k$ be a unit quaternion. Then, the matrix representation M of linear mapping φ can be written as

$$\begin{bmatrix} a_0^2+a_1^2-a_2^2-a_3^2 & -2a_0a_3+2a_1a_2 & 2a_0a_2+2a_1a_3\\ 2a_0a_3+2a_1a_2 & a_0^2+a_2^2-a_1^2-a_3^2 & 2a_2a_3-2a_0a_1\\ 2a_1a_3-a_0a_2 & 2a_0a_1+2a_2a_3 & a_0^2+a_3^2-a_2^2-a_1^2 \end{bmatrix}. \tag{6}$$

M is orthogonal since $MM^T=I$ and $detM=1$. So we can check that the linear mapping φ is a rotation in 3-dimensional space $\mathbb{R}^3$ (Ward, 1997).

Let $f:I\longrightarrow\mathbb{S}^2$ be a unit speed spherical curve with an arc length parameter t. $f'(t)$ is a unit tangent vector of $f(t)$. $f(t)\wedge f'(t)$ is the cross product of the position vector of curve $f(t)$ and the unit tangent vector $f'(t)$. The set $\{f'(t), f(t)\wedge f'(t), f(t)\}$ is an orthonormal frame along $f(t)$. This frame is called the Sabban frame of $f(t)$. Then we have the following spherical Frenet formulae of $f(t)$ on $\mathbb{S}^2$ as

$$\begin{aligned} f''(t)&=k_g(t)f(t)\wedge f'(t)-f(t),\\ (f(t)\wedge f'(t))'&=-k_g(t)f'(t),\\ f'(t)&=f'(t), \end{aligned} \tag{7}$$

where $k_g(t)$ is called the geodesic curvature of $f(t)$ on $\mathbb{S}^2$ which is $k_g(t)=\langle f''(t), f(t)\wedge f'(t)\rangle$ (Koenderink, 1990).

Theorem 1. Let $x:S\longrightarrow\mathbb{E}^3$ be a surface immersed in the 3-dimensional Euclidean space $\mathbb{E}^3$. Then S is a GCR surface if and only if it can be parameterized by

$$x(s,t) = s\left(\cos u(s) f(t) + \sin u(s) f(t) \wedge f'(t)\right), \quad (8)$$

where $u(s) = \int \frac{\cot\theta(s)}{s} ds, \theta(s) \notin \left\{0, \frac{\pi}{2}\right\}$ is the angle function and f is a unit speed curve on $\mathbb{S}^2$ (Fu & Munteanu, 2014).

3. New approach

By using the unit tangent vector of f, we can define the unit quaternion function as $q(s,t) = \cos u(s) - \sin u(s) f'(t)$, where $f'(t) = (f_1'(t), f_2'(t), f_3'(t))$. Since the unit quaternions $q(s,t)$ can be used in the linear mapping φ, we can write the matrix representation M of φ by using (2.5), as

$$\begin{bmatrix} \cos^2 u + \sin^2 u(f_1'^2 - f_2'^2 - f_3'^2) & 2\sin u(\cos u f_3' + \sin u f_1' f_2') & 2\sin u(\sin u f_1' f_3' - \cos u f_2') \\ 2\sin u(\sin u f_1' f_2' - \cos u f_3') & \cos^2 u - \sin^2 u(f_1'^2 - f_2'^2 + f_3'^2) & 2\sin u(\sin u f_2' f_3' + \cos u f_1') \\ 2\sin u(\sin u f_1' f_3' + \cos u f_2') & 2\sin u(\sin u f_2' f_3' - \cos u f_1') & \cos^2 u - \sin^2 u(f_1'^2 + f_2'^2 - f_3'^2) \end{bmatrix}. \quad (9)$$

For $f'(t)$ is the rotation axis of M,

$$f'(t) = M f'(t). \quad (10)$$

We can give the following theorem related to our study.

Theorem 2. Let $x : S \longrightarrow \mathbb{E}^3$ be a GCR surface immersed in the 3-dimensional Euclidean space $\mathbb{E}^3$ and $q(s,t) = \cos u(s) - \sin u(s) f'(t)$ be a unit quaternion, where $u(s) = \int \frac{\cot\theta(s)}{s} ds$. $q_1(s,t) = s f(t)$ is a pure quaternion and a surface in $\mathbb{E}^3$. Then the GCR surface S can be obtained by using quaternion product $\times$ as

$$x(s,t) = q(s,t) \times q_1(s,t). \quad (11)$$

Proof. If we use (3) for $q(s,t) = \cos u(s) - \sin u(s) f'(t)$ and $q_1(s,t) = s f(t)$, we have

$$\begin{aligned} q(s,t) \times q_1(s,t) &= s\cos u(s) f(t) - s\sin u(s) f'(t) \wedge f(t) \\ &= s(\cos u(s) f(t) + \sin u(s) f(t) \wedge f'(t)). \end{aligned} \quad (12)$$

So we can find the GCR surface as

$$x(s,t) = q(s,t) \times q_1(s,t). \quad (13)$$

This completes the proof.

Theorem 3. Let $x : S \longrightarrow \mathbb{E}^3$ be a GCR surface immersed in the 3-dimensional Euclidean space $\mathbb{E}^3$ and $q(s,t) = \cos u(s) - \sin u(s) f'(t)$ be a unit quaternion, where $u(s) = \int \frac{\cot\theta(s)}{s} ds$. $q_1(s,t) = s f(t)$ is a pure quaternion and a surface in $\mathbb{E}^3$. The every position vector of $x(s,t)$ is a pure quaternion and a GCR surface in $\mathbb{E}^3$. The quaternion product $q^n(s,t) \times x(s,t)$ is a GCR surface and $q^n(s,t) \times x(s,t) = q^{(n+1)}(s,t) \times q_1(s,t)$, where $n \in \mathbb{R}$.

Proof. If we use (3) for $q^n(s,t) = \cos(nu) - \sin(nu) f'(t)$ and $x(s,t) = s(\cos u f + \sin u f \wedge f')$, we get

$$\begin{aligned} q^n(s,t) \times x(s,t) &= s\cos(nu)\cos u f(t) \\ &\quad + \cos(nu)\sin u f(t) \wedge f'(t) \\ &\quad - s\sin(nu)\cos u f'(t) \wedge f(t) \\ &\quad - s\sin(nu)\sin u f'(t) \left(f(t) \wedge f'(t)\right) \end{aligned} \quad (14)$$

By using the Sabban frame of $f(t)$, we have

$$f'(t) \wedge f(t) = -f(t) \wedge f'(t), \; f'(t) \wedge (f(t) \wedge f'(t)) = f(t). \quad (15)$$

If we substitute (15) into (14), we get

$$\begin{aligned} q^n(s,t) \times x(s,t) &= s\left(\cos(nu)\cos u - \sin(nu)\sin u\right) f(t) \\ &\quad + s\left(\cos(nu)\sin u + \sin(nu)\cos u\right) f(t) \wedge f'(t) \\ &= s\left(\cos((n+1)u) f(t) + \sin((n+1)u) f(t) \wedge f'(t)\right). \end{aligned} \quad (16)$$

So it is not difficult to check that (16) is a GCR surface.

$$\begin{aligned} q^n(s,t) \times x(s,t) &= q^n(s,t) \times q(s,t) \times q_1(s,t) \\ &= q^{(n+1)}(s,t) \times q_1(s,t). \end{aligned} \quad (17)$$

This completes the proof.

So we can give the following corollaries.

Corollary 1. Let M be the matrix representation of the linear map φ for the unitary quaternion $q(s,t)$. So, the GCR surfaces can be obtained by the position vectors of $x(s,t)$ and the rotation matrix M as

$$M x(s,t), \quad (18)$$

where $x(s,t)$ is the GCR surface and pure quaternion.

Corollary 2. The GCR surface $x(s,t)$ can be written by the homothetic motion $q_2(s,t) = s q(s,t)$ as

$$\begin{aligned} x(s,t) &= q(s,t) \times q_1(s,t), \\ &= q_2(s,t) \times f(t). \end{aligned} \quad (19)$$

Also, the GCR surface is expressed by orthogonal matrix M as

$$\begin{aligned} x(s,t) &= Mq_1(s,t), \\ &= sMf(t). \end{aligned} \tag{20}$$

So we can give the following three remarks.

Remark 1. The unitary quaternion $q(s,t)$ rotates the position vectors and the points of the surface $x(s,t)$ and $q_1(s,t)$ through the angle $u(s)$ about the vector $f'(t)$.

Remark 2. We can conclude from Theorem 3 that if the GCR surface is rotated by the unit quaternion $q(s,t)$, we obtain the GCR surfaces. So we can say that the rotation of the GCR surfaces by unit quaternion $q(s,t)$ are invariant. Moreover, we can conclude from Corollary 1 that the GCR surfaces under linear mapping φ is invariant.

Remark 3. The GCR surface can be obtained by rotating of the position vector of the curve $f(t)$ with the homothetic motion $q_2(s,t)$ through the angle $u(s)$ about the vector $f'(t)$ and extending the homothetic scale s.

Example 1. If $p(t) = 2\arctan e^t$, then $f(t) = (\cos p(t)\cos t, \cos p(t)\sin t, \sin p(t))$ is a unit speed curve on $\mathbb{S}^2$ (Fu & Munteanu, 2014). We can give the Sabban frame vectors as

$$\begin{aligned} f(t) &= (\cos p(t)\cos t, \cos p(t)\sin t, \sin p(t)), \\ f'(t) &= (-p'\sin p\cos t - \cos p\sin t, -p'\sin p\sin t + \cos p\cos t, p'\cos p), \\ f(t)\times f'(t) &= (p'\sin t - \cos p\sin p\cos t, -p'\cos t - \cos p\sin p\sin t, \cos^2 p). \end{aligned} \tag{21}$$

So (8) can be written as

$$\begin{aligned} x(s,t) = (&s\cos u\cos p\cos t + sp'\sin u\sin t - s\sin u\cos p\sin p\cos t, \\ &s\cos u\cos p\sin t - sp'\sin u\cos t - s\sin u\cos p\sin p\sin t, \\ &s\cos u\sin p + s\sin u\cos^2 p). \end{aligned} \tag{22}$$

By using (17), for $n = 1$, we get the GCR surface as

$$\begin{aligned} q(s,t)\times x(s,t) = (&s\cos 2u\cos p\cos t + sp'\sin 2u\sin t - s\sin 2u\cos p\sin p\cos t, \\ &s\cos 2u\cos p\sin t - sp'\sin 2u\cos t - s\sin 2u\cos p\sin p\sin t, \\ &s\cos 2u\sin p + s\sin 2u\cos^2 p). \end{aligned} \tag{23}$$

For $u(s) = s$, $p(t) = (2\arctan e^t)$ and $p'(t) = (\sin(2\arctan e^t))$, we can draw Figure 1.

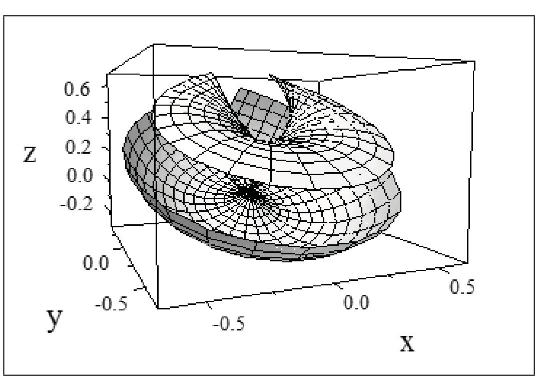

Fig. 1. The GCR surface for $u(s) = s$

Example 2. For $f(t) = (\frac{1}{3}\cos 3t, \frac{1}{3}\sin 3t, \frac{2\sqrt{2}}{3})$, we can give the Sabban frame vectors as

$$\begin{aligned} f(t) &= (\frac{1}{3}\cos 3t, \frac{1}{3}\sin 3t, \frac{2\sqrt{2}}{3}), \\ f'(t) &= (-\sin 3t, \cos 3t, 0), \\ f(t)\times f'(t) &= (-\frac{2\sqrt{2}}{3}\cos 3t, -\frac{2\sqrt{2}}{3}\sin 3t, \frac{1}{3}). \end{aligned} \tag{24}$$

We have $q(s,t) = \cos u(s) - \sin u(s) f'(t)$ and $q_1(s,t) = sf(t)$. Then, by using (20), we get the GCR surface $x(s,t)$ as

$$s\begin{bmatrix}\cos^2 u+\sin^2 u(\sin^2 3t-\cos^2 3t) & -2\sin^2 u\sin 3t\cos 3t & -2\sin u\cos u\cos 3t\\ -2\sin^2 u\sin 3t\cos 3t & \cos^2 u-\sin^2 u(\sin^2 3t-\cos^2 3t)) & -2\sin u\cos u\sin 3t\\ 2\sin u\cos u\cos 3t & 2\sin u\cos u\sin 3t & \cos^2 u-\sin^2 u\end{bmatrix}$$

$$.\begin{bmatrix}\frac{1}{3}\cos 3t\\ \frac{1}{3}\sin 3t\\ \frac{2\sqrt[2]{2}}{3}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{3}s\cos 2u\cos 3t-\frac{2\sqrt{2}}{3}s\sin 2u\cos 3t\\ \frac{1}{3}s\cos 2u\sin 3t-\frac{2\sqrt{2}}{3}s\sin 2u\sin 3t\\ \frac{2\sqrt{2}}{3}s\cos 2u+\frac{1}{3}s\sin 2u\end{bmatrix}.$$

For $u(s)=\dfrac{1}{2s}$, we can draw Figure 2.

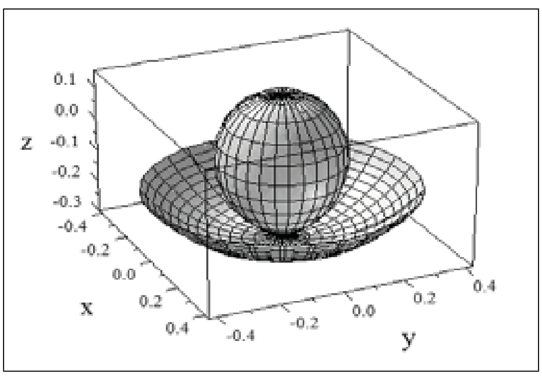

Fig. 2. The GCR surface for $u(s)=\dfrac{1}{2s}$

4. Conclusions

In the present of this research, we have given a special unit quaternion. Then, we show that the GCR surfaces can be obtained by the quaternion product of the unit quaternion and by its matrix representation. Also, it has been explained that the GCR surfaces are invariant under the rotations by the unit quaternions. Finally, it has been given some results and examples.

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خلاصة

نثبت في هذه الدراسة أن السطوح من النوع GCR يمكن الحصول عليها بواسطة جداء مرباعي لوحدة مرباعية و مرباعية بحتة. كما نرى أيضاً أن الجداء المرباعي لوحدة مرباعية و متجه الموضع للسطح من النوع GCR بحت هو نفسه سطح من النوع GCR. نعطي بعد ذلك بعض النتائج حول السطوح من النوع GCR. وذلك بإستخدام المرباعيات.