Wormhole solutions and energy conditions in $f(R,T)$ gravity with exponential models

G. Mustafa[1*], M. Farasat Shamir[2], Anum Fazal[2]
[1]*Dept. of Physics, Zhejiang Normal University, Jinhua, People's Republic of China*
[2]*National University of Computer and Emerging Sciences, Lahore Campus, Pakistan.*
**Corresponding author: gmustafa3828@gmail.com*

Abstract

This study explores the new exact solutions of wormhole geometry by imposing the inconsistent Ricci scalar via inhomogeneous spacetime. The current analysis is dealing with the modified $f(R,T)$ theory of gravity. Two different models of gravity that are $f_1(R) = R - \alpha\gamma(1 - e^{-\frac{R}{\gamma}})$ known as exponential gravity model and $f_1(R) = R - \alpha\gamma tanh(\frac{R}{\gamma})$ known as Tsujikawa model, where α, γ are model parameters with matter coupling are considered for the current study. The new feasible solutions for these models by comparing normal and inhomogeneous spacetimes is calculated. Further, we discuss the different properties of the obtained wormhole solutions by taking suitable values of the model parameters analytically and graphically. Moreover, we consider a specific shape functions i.e., $b(r) = r_0 \log(\frac{r}{r_0} + 1)$ and discuss the energy conditions for both models. The presented wormhole solutions are physically acceptable for the considered exponential and Tsujikawa gravity.

Keywords: Exponential gravity; exotic matter; inhomogeneous spacetime; wormholes; $f(R,T)$ theory of gravity.

1. Introduction

Einstein's general relativity (GR) has been served as the most successful theory of gravitation to explain and understand various mysteries of the astrophysical as well as the cosmological realm. A wormhole is a theoretical connection between remote regions of the universe, reducing travelling time and distance. The wormhole concept has an early history starting with Flamm (1916), who constructed the Schwarzschild solution of field equations as a non-traversable wormhole. Einstein-Rosen bridge proposed (Einstein *et al.*, 1935) the existence of a bridge, which used to join two copies of Schwarzschild spacetime for which the wormhole throat implodes, thus forming a singularity. The topological structure (Ellis, 1973) introduced the traversable wormhole concept by coupling geometry and scalar field, creating a geodesically complete manifold with no horizon. Bronnikov (1973) explored the scalar-electro-vacuum configurations without scalar charge. Clement (1984) gave a class of traversable wormholes in higher dimensions. The wormhole geometry (Morris *et al.*, 1988) proposed the idea of a traversable wormhole by joining two distant cosmic regions (asymptotically flat) by a throat supported by an exotic matter violating the null energy condition (NEC) that keeps the wormhole throat open. The physical viability of wormhole configuration requires confining this matter's usage, which is controversial. There has been extensive work on the construction of wormholes from black hole spacetimes and analysis of their various physical aspects (Richarte *et al.*, 2007., Eiroa *et al.*, 2008., Sharif *et al.*, 2016., Övgun, 2018., Falco *et al.,* 2020).

In curvature, the occurrence of higher-order terms would be feasible for constructing wormholes with thin shells detained by ordinary matter in the framework of the modified theory of gravity (Mazharimousavi *et al.*, 2010, Mazharimousavi *et al.*, 2011). In recent times, solutions of the wormholes have been studied within the context of modified gravity, for example, Kaluza-Klein gravity (Dzhunushaliev *et al.*, 2011), the theory of Einstein-Gauss-Bonnet (Mehdizadeh *et al.*, 2015, Zangeneh *et al.*, 2015), theory of Einstein-Cartan (Bronnikov *et al.*, 2015, Bronnikov *et al.*, 2016, Mehdizadeh *et al.,* 2017), Brans-Dicke theory (Agnese *et al.*, 1995, Nandi *et al.*, 1997, Lobo *et al.*, 2010, Sushkov *et al.,* 2011), scalar-tensor gravity (Shaikh *et al.,* 2016) and Born-Infeld theory (Eiroa *et al.*, 2012). Recently, modified gravity theories have been considered relating to cosmological matters such as gravastars, wormholes, black holes and strange stars. These theories explain the dark energy problems and can describe the accelerating extension of the universe (Deffayet *et al.*, 2002, Carrol *et al.*, 2004, Nojiri *et al.*, 2003). One such theory with the modified form of Einstein gravity is $f(R,T)$ gravity theory, where R represents Ricci scalar, and T represents the trace of energy-momentum tensor (Harko *et al.*, 2011).

Some captivating cosmological $f(R,T)$ representations are auxiliary scalar field, models of dark matter and models of the anisotropic universe, which has been established using various setups (Houndjo *et al.*, 2012, Jamil *et al.*, 2012). Diverse cosmological uses of the $f(R,T)$ theory of gravity have been given in texts such as thermodynamics, compact stars, phase space perturbations and constancy of collapsing matter (Singh *et al.*, 2014, Shabani *et al.*, 2013, Shabani *et al.*, 2014, Santos *et al.*, 2013, Alvarenga *et al.*, 2013, Baffou *et al.*, 2015). Moraes *et al.,* (2017) obtained general analytic explanations for static wormholes in the $f(R,T)$ theory of gravity. Recently a non-linear $f(R,T)$ function has been defined. The spherical areas where energy conditions are fulfilled for traversable static wormholes have been explored by Godani and Samanta (2019). Shamir *et al.,* (2021) used non-commutative geometry to investigate the wormhole solutions in $f(R,T)$ gravity by considering Gaussian and Lorentzian sources. For the derivation of cosmological forces in various settings, numerous useful procedures for examining the $f(R,T)$ theory of gravity have been done. The split-up scenario which is $f(R,T) = f_1(R) + f_2(T)$ is considered in most cases because of its simplicity and also one can search the impact from T without stipulating $f_1(R)$ and similarly the impact from R without stipulating $f_2(T)$. Such reformation of the $f(R,T)$ theory of gravity is examined (Houndjo, 2012).

In the present paper, we want to find the wormhole solutions using a non-constant Ricci scalar in the framework of the feasible $f(R,T)$ theory of gravity. Inspired by the discussion mentioned above, we study static spherically symmetric wormholes in the $f(R,T)$ theory of gravity background in this article. Following is the organization of this paper: Section II includes the description of the basic formalism and corresponding spherically symmetric, static spacetime in $f(R,T)$ gravity.The discussion of energy conditions is provided in Section III. In Section IV, traversable wormhole solutions with inhomogeneous spacetime have been studied using both exponential gravity and the Tsujikawa model. The entire section V is dedicated to the study of energy conditions using $b(r) = r_0 \log(\frac{r}{r_0} + 1)$. And a summary of the whole study is highlighted in the last section.

2. $f(R,T)$ gravity Wormholes

The action of $f(R,T)$ gravity is as follows

$$S = \frac{1}{16\pi} \int d^4x f(R,T)\sqrt{-g} + \int \mathcal{L}_m d^4x \sqrt{-g}, \tag{1}$$

with R being Ricci scalar and T being trace of energy momentum tensor by T in an random $f(R,T)$ function, $\mathcal{L}_m$ being Lagrangian density and metric determinant is g. Variation of Eq. (1) with metric tensor gives the following field equations (Cognola, 2008)

$$f_R(R,T)R_{\gamma\xi} - \frac{1}{2}f(R,T)g_{\gamma\xi} + (g_{\gamma\xi}\Box - \nabla_\gamma\nabla_\xi)f_R(R,T) = 8\pi(T_{\gamma\xi}) - f_T(R,T)T_{\gamma\xi} - f_T(R,T)\Theta_{\gamma\xi}. \tag{2}$$

By contracting Eq. (2) with $g^{\gamma\xi}$, T and R are found to have a new relation such that

$$f_R(R,T)R + 3\Box f_R(R,T) - 2f(R,T) = 8\pi T - f_T(R,T)T - f_T(R,T)\Theta. \tag{3}$$

Covariant derivative has been denoted as ∇ whereas $\Box$ denotes d'Alembert operator. Also,

$$f_R(R,T) = \frac{\partial f(R,T)}{\partial R}, \qquad f_T(R,T) = \frac{\partial f(R,T)}{\partial T}, \qquad \Theta_{\gamma\xi} = g_{\gamma\xi}\frac{\partial T_{\gamma\xi}}{\partial g^{\gamma\xi}}. \tag{4}$$

Here, we will take anisotropic fluid for energy momentum tensor as

$$T_{\gamma\xi} = (\rho + p_t)\mathcal{V}_\gamma \mathcal{V}_\xi - p_t g_{\gamma\xi} + (p_r - p_t)\chi_\gamma \chi_\xi, \tag{5}$$

$\mathcal{V}_\gamma$ and χ_γ indicates four velocity vectors of the fluid with $\mathcal{V}^\gamma = e^{-\alpha}\delta_0^\gamma$ and $\chi^\gamma = e^{-\beta}\delta_1^\gamma$, hence fulfilling the relations $\mathcal{V}^\gamma \mathcal{V}_\gamma = -\chi_\gamma^\gamma = 1$. By selecting $\mathcal{L}_m = \rho$, we get

$$\Theta_{\gamma\xi} = -2T_{\gamma\xi} - \rho g_{\gamma\xi}. \tag{6}$$

With Eqs. (2), (3) and (6), we attain modified field equations in the given form

$$\begin{aligned} f_R(R,T)G_{\gamma\xi} &= (8\pi + f_T(R,T))T_{\gamma\xi} + \Big(\nabla_\gamma \nabla_\xi f_R(R,T) - \frac{1}{4} g_{\gamma\xi}(8\pi + f_T(R,T))T \\ &+ \Box f_R(R,T) + f_R(R,T)R\Big). \end{aligned} \tag{7}$$

In spherically symmetric spacetime, geometry of wormhole is as follows

$$ds^2 = -e^{2\psi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{8}$$

here $b(r)$ and $\psi(r)$ are radial coordinate functions with $b(r)$ being shape function and $\psi(r)$ being redshift function. In the present work, we take constant red shift function i.e. $\psi'(r) = 0$. Now by putting values in Equation (7) with metric given in Equation (8), we get

$$\frac{b'}{r^2} = \frac{(8\pi + f_T(R,T))}{f_R(R,T)}\rho + \frac{X}{f_R(R,T)}, \tag{9}$$

$$\begin{aligned} -\frac{b}{r^3} &= \frac{(8\pi + f_T(R,T))}{f_R(R,T)}p_r + \frac{1}{f_R(R,T)}(1 - \frac{b}{r})[(f''_R(R,T) - f'_R(R,T)\frac{b'r - F_s}{2r^2(1 - b/r)})] \\ &- \frac{X}{f_R(R,T)}, \end{aligned} \tag{10}$$

$$-\frac{b'r - b}{2r^3} = \frac{8\pi + f_T(R,T)}{f_R(R,T)}p_t + \frac{1}{f_R(R,T)}(1 - \frac{b}{r})\frac{f'_R(R,T)}{r} - \frac{X}{f_R(R,T)}, \tag{11}$$

where

$$X \equiv X(r) = \frac{1}{4}(f_R(R,T)R + \Box f_R(R,T) + (8\pi + f_T(R,T))T). \tag{12}$$

Ricci Scalar for spacetime given in Equation(8) is

$$R = \frac{2b'}{r^2}, \tag{13}$$

and

$$\Box f_R(R,T) = (1 - \frac{b}{r})\left[f''_R(R,T) - f'_R(R,T)\frac{b'r - b}{2r^2(1 - \frac{b}{r}) + \frac{2f'_R(R,T)}{r}}\right]. \tag{14}$$

The above-mentioned system of equations is not easy to solve for ρ, p_r and p_t as it has higher order derivatives with several unknowns. To make these calculations easy, we choose the split-up scenario which is $f(R,T) = f_1(R) + f_2(T)$. Taking $f_2(T) = \lambda T$ with λ being coupling parameter. Then putting in the form of $f(R,T)$ and making calculations in Eqs. (9-11) a little easier to solve. We obtain

$$\rho = \frac{b' f_R}{r^2(8\pi + \lambda)}, \tag{15}$$

$$p_r = -\frac{b f_R}{r^3(8\pi + \lambda)} + \frac{f'_R}{2r^2(8\pi + \lambda)}(b'r - F_s) - (1 - \frac{b}{r})\frac{f''_R}{8\pi + \lambda}, \tag{16}$$

$$p_t = -\frac{f'_R}{r(8\pi + \lambda)}(1 - \frac{b}{r}) + \frac{f_R}{2r^3(8\pi + \lambda)}(b'r - b). \tag{17}$$

3. Energy Conditions

Null energy condition (NEC), weak energy condition (WEC), strong energy condition (SEC) and dominant energy condition (DEC) are main energy conditions. Aforementioned energy conditions are defined as

$$\begin{aligned} NEC &\Leftrightarrow T_{\gamma\xi}k^{\gamma}k^{\xi} \geq 0, & WEC \Leftrightarrow T_{\gamma\xi}V^{\gamma}V^{\xi} \geq 0, \\ SEC &\Leftrightarrow (T_{\gamma\xi} - \frac{T}{2}g_{\gamma\xi})V^{\gamma}V^{\xi} \geq 0, & DEC \Leftrightarrow T_{\gamma\xi}V^{\gamma}V^{\xi} \geq 0, \end{aligned}$$

where k^{γ} is null vector and V^{γ} is timelike vector. For DEC, $T_{\gamma\xi}V^{\gamma}$ is not space like. The following energy conditions with regards to principal pressure are defined as

$$\begin{aligned} NEC &\Leftrightarrow \forall j,\ \rho + p_j \geq 0, & WEC \Leftrightarrow \rho \geq 0\ and\ \forall j,\ \rho + p_j \geq 0, \\ SEC &\Leftrightarrow \forall j,\ \rho + p_j \geq 0,\ \rho + \sum_j p_j \geq 0, & DEC \Leftrightarrow \rho \geq 0\ and\ \forall j,\ p_j \epsilon [-\rho, +\rho]. \end{aligned}$$

Here we take these conditions with regards to principal pressures which is as follows

$$\begin{aligned} NEC &: \rho + p_r \geq 0, \quad \rho + p_t \geq 0, \\ WEC &: \rho \geq 0, \quad \rho + p_r \geq 0, \quad \rho + p_t \geq 0, \\ SEC &: \rho + p - r \geq 0, \quad \rho + p_t \geq 0, \quad \rho + p_r + 2p_t \geq 0, \\ DEC &: \rho \geq 0, \quad \rho - |p_r| \geq 0, \quad \rho - |p_t| \geq 0. \end{aligned}$$

Theses energy conditions are fulfilled by normal matter because of positive density and positive pressure. Einstein's field theory tells us that wormholes are full of exotic matter which is not similar to normal matter.

4. Traversable Wormhole Solutions with Inhomogeneous Spacetime

Since, inhomogeneous spacetime (Golchina & Mehdizadeh, 2019) merges easily with cosmological background so we study the Ricci scalar of wormhole geometry as

$$R = 6a_1 + \frac{6a_2}{r^n}, \tag{18}$$

with a_1, a_2 and n being free parameters. Now we have to find wormhole solutions with Ricci scalar given in equation (18). Shape function can be found by comparing equation (13) with equation (18),

$$b(r) = r^3(a_1 + \frac{3a_2}{3-n}r^{-n}) + C, \tag{19}$$

with C being constant of integration. In this work, we take $C = 0$. Following conditions should be satisfied by the shape function $b(r)$ for wormhole solution:

$$\begin{aligned} i) &\quad b(r_0) = r_0, \\ ii) &\quad b'(r_0) < 1, \\ iii) &\quad 1 - \frac{b(r)}{r} > 0. \end{aligned} \tag{20}$$

We can find the value of a_2 by putting condition $i)$ in Equation (19) as

$$a_2 = \frac{n-3}{3}r_0^{n-2}(a_1 r_0^2 - 1). \tag{21}$$

By inserting Equation (21) in Equation (19), shape function is found as

$$b(r) = (r_0^{n-2} - r_0^n a_1)r^{3-n} + a_1 r^3. \tag{22}$$

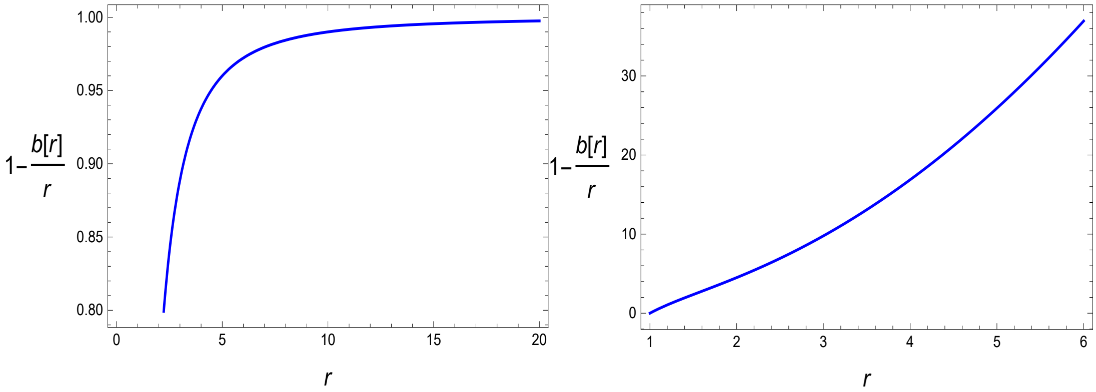

Fig. 1: Shows the asymptotically flat $a_1 = 0$ and asymptotically hyperbolic $a_1 = -1$ wormhole solutions respectively with $n = 4$ and $r_0 = 1$.

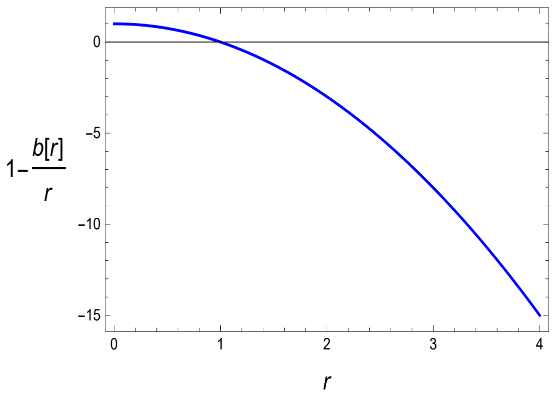

Fig. 2: Shows the asymptotically spherical $a_1 = 1$ wormhole solution with $n = 4$ and $r_0 = 1$.

The shape function for $a_1 = 0, \pm 1$ are shown in Figure 1 and Figure 2. We can also write Equation (22) as

$$\frac{b(r)}{r} = a_1 r^2 + (r_0^{n-2} - a_1 r_0^n) r^{2-n}. \tag{23}$$

Because of the condition $ii)$ given in Equation (20) we should put $n > 2$ in Equation (23). Due to the metric (Equation (8)), we deduce that the solutions obtained with $a_1 = 0$, $a_1 = 1$ and $a_1 = -1$ at large value of r matches the flat, spherical and hyperbolic Friedmann-Robertson-Walker (FRW) universe respectively. Solutions of wormholes can be obtained for the shape function (22) by using Equations (15-17).

$$\rho = \frac{f_R}{r^2(8\pi+\lambda)}(3a_1 r^2 + r^{2-n}(3-n)(r_0^{n-2} - a_1 r_0^n)), \tag{24}$$

$$p_r = -\frac{f_R''}{8\pi+\lambda} + \frac{f_R'}{2r(8\pi+\lambda)}(3a_1 r^2 + r^{2-n}(3-n)(r_0^{n-2} - a_1 r_0^n)) - \frac{f_r A}{r^2} - \frac{f_R' A}{2r} + f_R'' A, \tag{25}$$

$$p_t = -\frac{f_R'}{r(8\pi+\lambda)} - \frac{f_R}{2r^2(8\pi+\lambda)}(3a_1 r^2 + r^{2-n}(3-n)(r_0^{n-2} - a_1 r_0^n)) + \frac{f_R A}{2r^2} + \frac{f_R'}{r}, \tag{26}$$

where

$$A = \frac{a_1 r^3 + r^{3-n}(r_0^{n-2} - a_1 r_0^n)}{r(8\pi+\lambda)}.$$

Now we consider a captivating $f(R)$ gravity model which is exponential gravity model (Cognola, 2008., Elizalde, 2011.)

$$f_1(R) = R - \alpha\gamma(1 - e^{-\frac{R}{\gamma}}), \tag{27}$$

where α and γ are taken as free positive parameters. We get the following set of equations by using Equation (27) in Equations (24-26)

$$\rho = \frac{C}{r^2(8\pi+\lambda)}\left(1 + \frac{4\alpha C}{r^2} + 2^{n-1} n\gamma\left(\frac{C}{r^2}\right)^{n-1}\right), \tag{28}$$

$$\begin{aligned}
p_r = & -\frac{1}{r^3(8\pi+\lambda)}\Big(I\Big)\left(1 + \frac{4\alpha C}{r^2} + 2^{n-1} n\gamma\left(\frac{C}{r^2}\right)^{n-1}\right) + F \\
& - \frac{1}{2r^2(8\pi+\lambda)}\Big(I\Big)\Big(G\Big) - \frac{1}{8\pi+\lambda}\Big(H\Big) + \frac{1}{r(8\pi+\lambda)}\Big(a_1 r^3 + r^{3-n}(r_0^{n-2} \\
& - a_1 r_0^n)\Big)\left(-\frac{16\alpha D}{r^3} + \frac{24\alpha C}{r^4} + \frac{4\alpha E}{r^2} + 2^{n-1}(n-2)(n-1)n\gamma\left(\frac{C}{r^2}\right)^{n-3}\left(\frac{D}{r^2} - \frac{2C}{r^3}\right)^2 \right.\\
& \left. + 2^{n-1}(n-1)n\gamma\left(\frac{C}{r^2}\right)^{n-2}\left(\frac{4D}{r^3} + \frac{6C}{r^4} + \frac{E}{r^2}\right)\right),
\end{aligned} \tag{29}$$

$$\begin{aligned}
p_t = & -\frac{C}{2r^2(8\pi+\lambda)}\left(1 + \frac{4\alpha C}{r^2} + 2^{n-1} n\gamma\left(\frac{C}{r^2}\right)^{n-1}\right) + \frac{1}{2r^3(8\pi+\lambda)}\Big(I\Big)\left(1 + \frac{4\alpha C}{r^2}\right.\\
& \left. + 2^{n-1} n\gamma\left(\frac{C}{r^2}\right)^{n-1}\right) - \frac{1}{r(8\pi+\lambda)}\Big(G\Big) + \frac{1}{r^2(8\pi+\lambda)} \\
& \times\Big(I\Big)\left(\frac{4\alpha D}{r^2} - \frac{8\alpha C}{r^3} + 2^{n-1}(n-1)n\gamma\left(\frac{C}{r^2}\right)^{n-2}\left(\frac{D}{r^2} - \frac{2C}{r^3}\right)\right),
\end{aligned} \tag{30}$$

where,

$$\begin{aligned}
C &= 3a_1 r^2 + r^{2-n}(3-n)(r_0^{n-2} - a_1 r_0^n), \\
D &= 6a_1 r + r^{1-n}(2-n)(3-n)(r_0^{n-2} - a_1 r_0^n), \\
E &= 6a_1 + r^{-n}(1-n)(2-n)(3-n)(r_0^{n-2} - a_1 r_0^n).
\end{aligned}$$

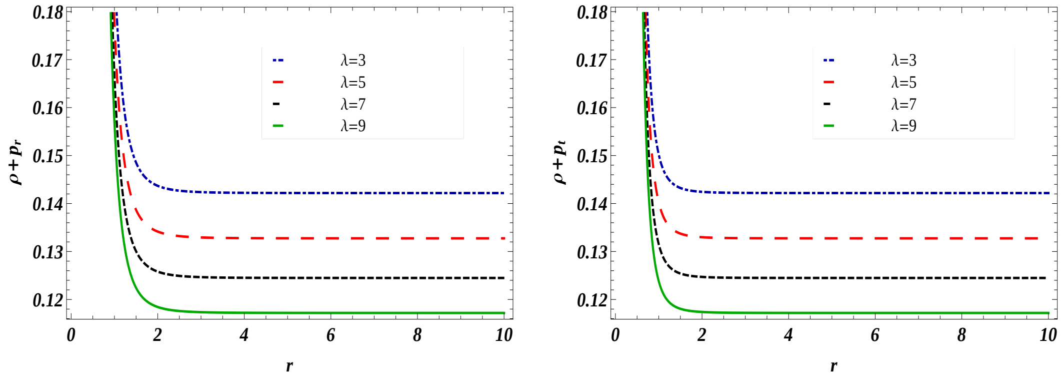

Fig. 3: Shows the development of $\rho + p_r$ and $\rho + p_t$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

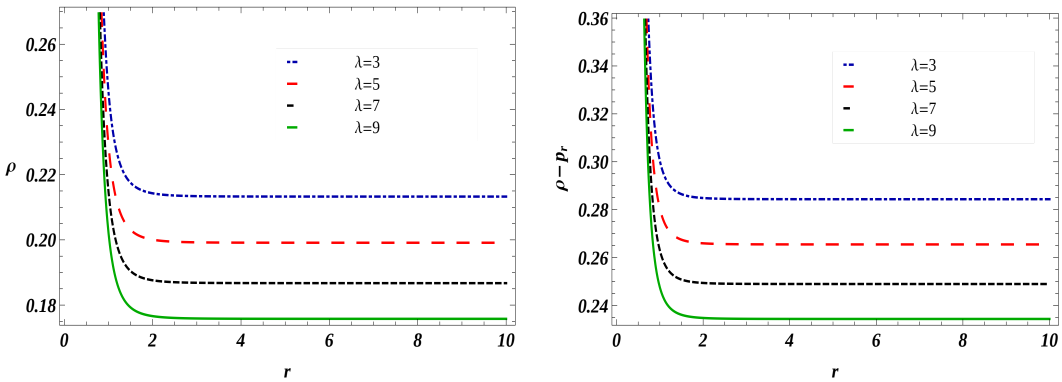

Fig. 4: Shows the development of ρ and $\rho - p_r$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

$$
\begin{aligned}
F &= \frac{C}{2r(8\pi+\lambda)}\left(\frac{4\alpha D}{r^2} - \frac{8\alpha C}{r^3} + 2^{n-1}(n-1)n\gamma\left(\frac{C}{r^2}\right)^{n-2}\left(\frac{D}{r^2} - \frac{2C}{r^3}\right)\right),\\
G &= \frac{4\alpha D}{r^2} - \frac{8\alpha C}{r^3} + 2^{n-1}(n-1)n\gamma\left(\frac{C}{r^2}\right)^{n-2}\left(\frac{D}{r^2} - \frac{2C}{r^3}\right),\\
H &= -\frac{16\alpha D}{r^3} + \frac{24\alpha C}{r^4} + \frac{4\alpha E}{r^2} + 2^{n-1}(n-2)(n-1)n\gamma\\
&\quad \times\left(\frac{C}{r^2}\right)^{n-3}\left(\frac{D}{r^2} - \frac{2C}{r^3}\right)^2 + 2^{n-1}(n-1)n\gamma\left(\frac{C}{r^2}\right)^{n-2}\left(-\frac{4D}{r^3} + \frac{6C}{r^4} + \frac{E}{r^2}\right),\\
I &= a_1 r^3 + r^{3-n}(r_0^{n-2} - a_1 r_0^n)
\end{aligned}
$$

Now, considering one more specific and interesting model for $f(R)$ theory of gravity i.e., Tsujikawa

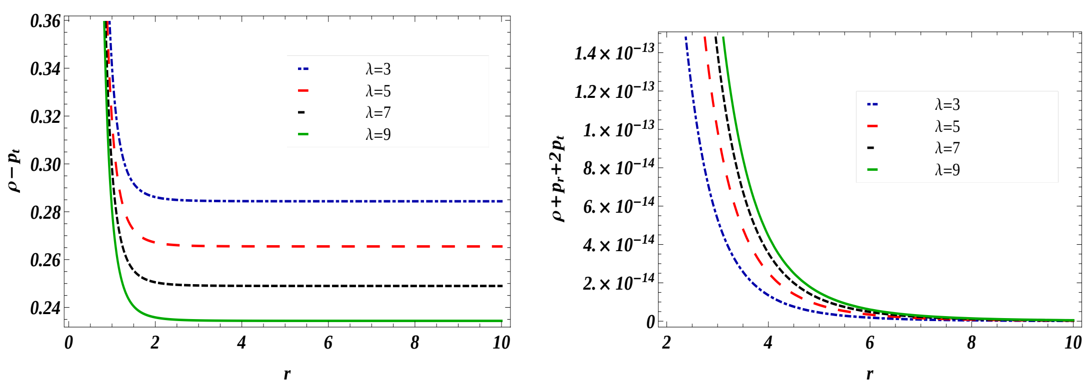

Fig. 5: Shows the development of $\rho - p_t$ and $\rho + p_r + 2p_t$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

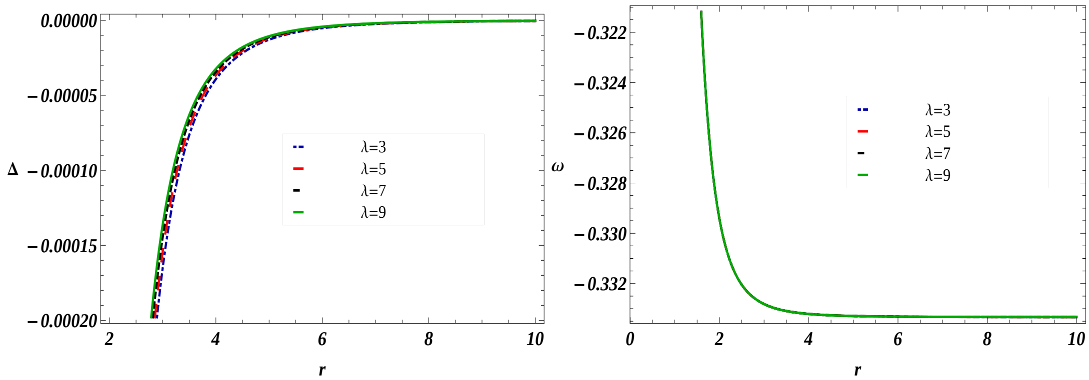

Fig. 6: Shows the development of Δ and ω with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

model and is represented as (Tsujikawa 2008., DeFelice 2010.)

$$f_1(R) = R - \alpha\gamma tanh(\frac{R}{\gamma}), \tag{31}$$

with α and γ being positive and free parameters of the given model. Now, by inserting the above model (31) in Equations (24-26), we obtain given set of equations:

$$\rho = \frac{C_1}{r^2(8\pi+\lambda)}(1-\alpha S^2), \tag{32}$$

$$\begin{aligned} p_r &= -\frac{1-\alpha S^2}{r^3(8\pi+\lambda)}\Big(I\Big) + \frac{\alpha C_1 S^2 T}{r(8\pi+\lambda)}\Big(\frac{2D_1}{r^2\gamma} - \frac{4C_1}{r^3\gamma}\Big) - \frac{\alpha S^2 T}{r^2(8\pi+\lambda)}\Big(I\Big)\Big(\frac{2D_1}{r^2\gamma} - \frac{4C_1}{r^3\gamma}\Big) \\ &- \frac{1}{8\pi+\lambda}\Bigg(2\alpha S^4\Big(\frac{2D_1}{r^2\gamma} - \frac{4C_1}{r^3\gamma}\Big)^2 + 2\alpha S^2 T\Big(\frac{-8D_1}{r^3\gamma} + \frac{12C_1}{r^4\gamma} + \frac{2E_1}{r^2\gamma}\Big) - 4\alpha S^2 T^2\Big(\frac{2D_1}{r^2\gamma} \quad (33) \\ &- \frac{4C_1}{r^3\gamma}\Big)^2\Bigg) + \frac{1}{r(8\pi+\lambda)}\Big(I\Big)\Bigg(2\alpha S^4\Big(\frac{2D_1}{r^2\gamma} - \frac{4C_1}{r^3\gamma}\Big)^2 + 2\alpha S^2 T\Big(-\frac{8D_1}{r^3\gamma} + \frac{12C_1}{r^4\gamma} + \frac{2E_1}{r^2\gamma}\Big) \\ &- 4\alpha S^2 T^2\Big(\frac{2D_1}{r^2\gamma} - \frac{4C_1}{r^3\gamma}\Big)^2\Bigg), \end{aligned} \tag{34}$$

$$\begin{aligned} p_t &= \frac{-C_1}{2r^2(8\pi+\lambda)}(1-\alpha S^2) + \frac{1}{2r^3(8\pi+\lambda)}\Big(I\Big)(1-\alpha S^2) - \frac{2\alpha S^2 T}{r(8\pi+\lambda)}\Big(\frac{2D_1}{r^2\gamma} - \frac{4C_1}{r^3\gamma}\Big) \\ &+ \frac{2\alpha S^2 T}{r^2(8\pi+\lambda)}\Big(I\Big)\Big(\frac{2D_1}{r^2\gamma} - \frac{4C_1}{r^3\gamma}\Big), \end{aligned} \tag{35}$$

where,

$$\begin{aligned} S &= sech(\frac{2C_1}{r^2\gamma}), \\ T &= tanh(\frac{2C_1}{r^2\gamma}), \end{aligned}$$

and

$$\begin{aligned} C_1 &= 3a_1 r^2 + r^{2-n}(3-n)(r_0^{n-2} - a_1 r_0^n), \\ D_1 &= 6a_1 r + r^{1-n}(2-n)(3-n)(r_0^{n-2} - a_1 r_0^n), \\ E_1 &= 6a_1 + r^{-n}(1-n)(2-n)(3-n)(r_0^{n-2} - a_1 r_0^n). \end{aligned}$$

Shape function plays a vital part in defining the character of a wormhole structure. Since for each value of radial coordinate r, the energy density is positive. As seen in figure 4 and Figure 8 graphs on the left side, the obtained energy density is non-negative and decreasing. NEC, WEC, and DEC except are

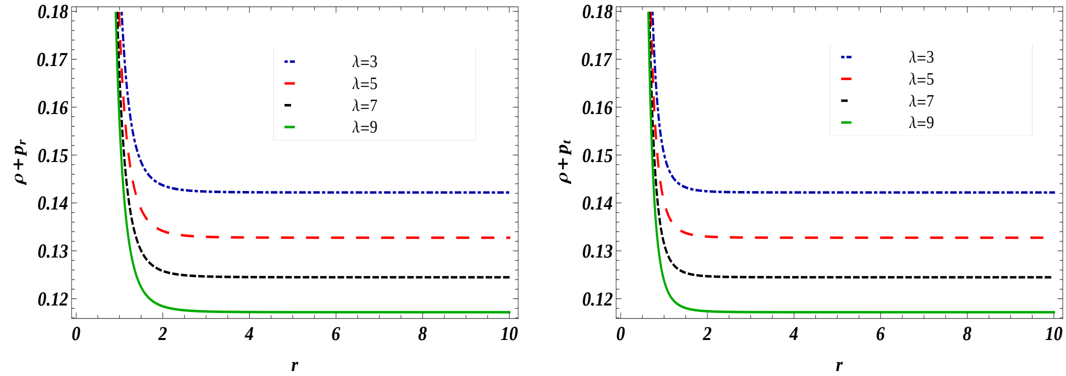

Fig. 7: Shows the development of $\rho + p_r$ and $\rho + p_t$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

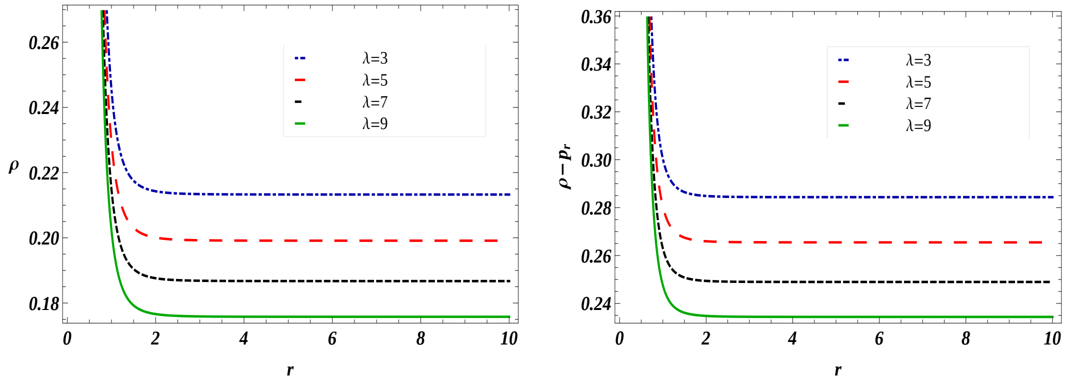

Fig. 8: Shows the development of ρ and $\rho - p_r$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

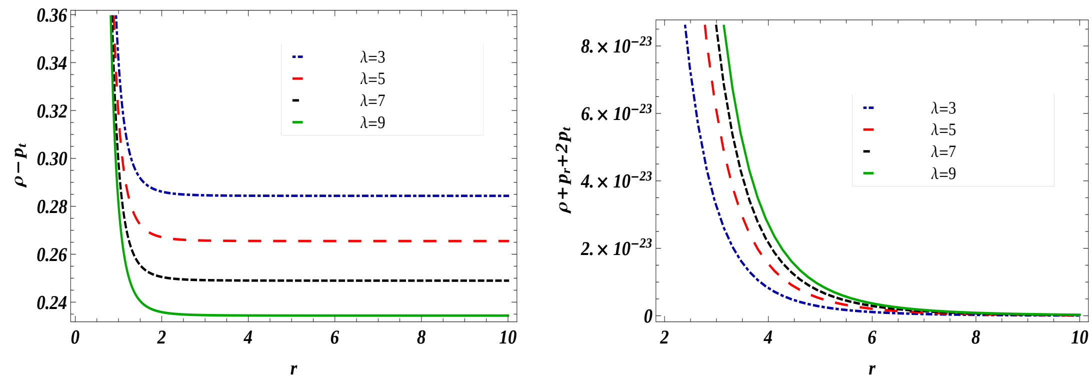

Fig. 9: Shows the development of $\rho - p_t$ and $\rho + p_r + 2p_t$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

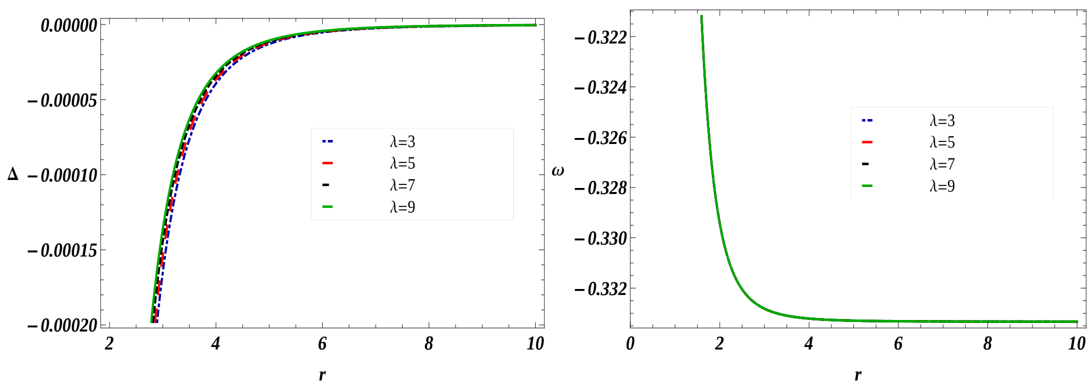

Fig. 10: Shows the development of Δ and ω with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

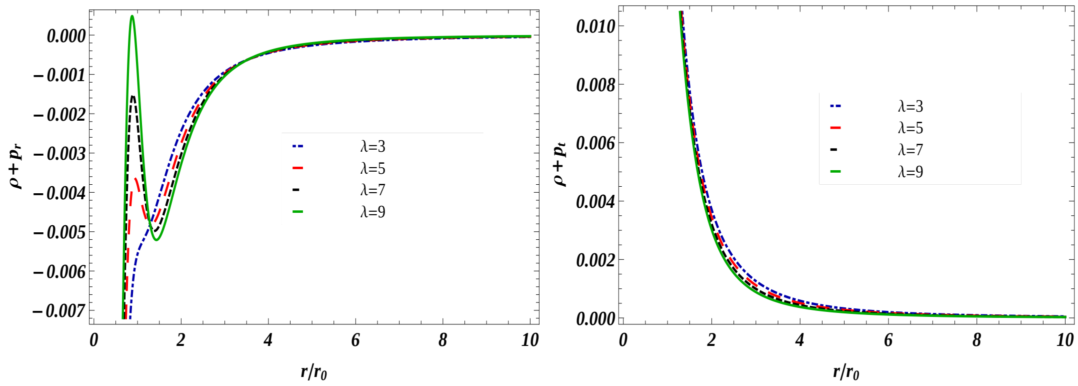

Fig. 11: Shows the development of $\rho + p_r$ and $\rho + p_t$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

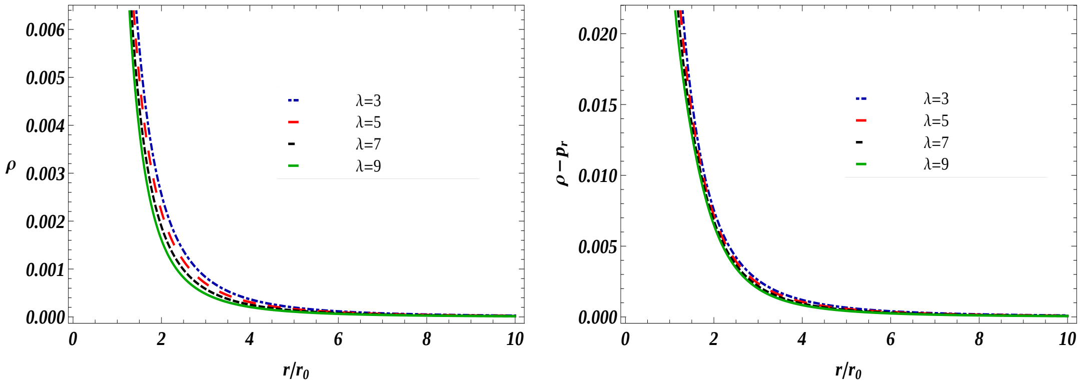

Fig. 12: Shows the development of ρ and $\rho - p_r$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

satisfied throughout but SEC is not see (figures 3-5, and figures 7-9). Anisotropy parameter is denoted as $\Delta = p_t - p_r$. The negative value of Δ represents the attractive geometry of the wormhole, whereas the positive value represents the repulsive nature of geometry. For the given shape function, both the models have a negative anisotropy parameter which shows the attractive nature of geometry inside the wormhole see (figures 6,10). Furthermore in terms of radial pressure, equation of state parameter is defined as $\omega = \frac{p_r}{\rho}$. It tells us the type of fluid filled in the wormhole's structure. For the given shape function, value of ω for both the models is between -1 and 0 that indicates the existence of non-phantom fluid see (figures 6,10). Hence, such solutions to wormholes can occur without the existence of exotic matter.

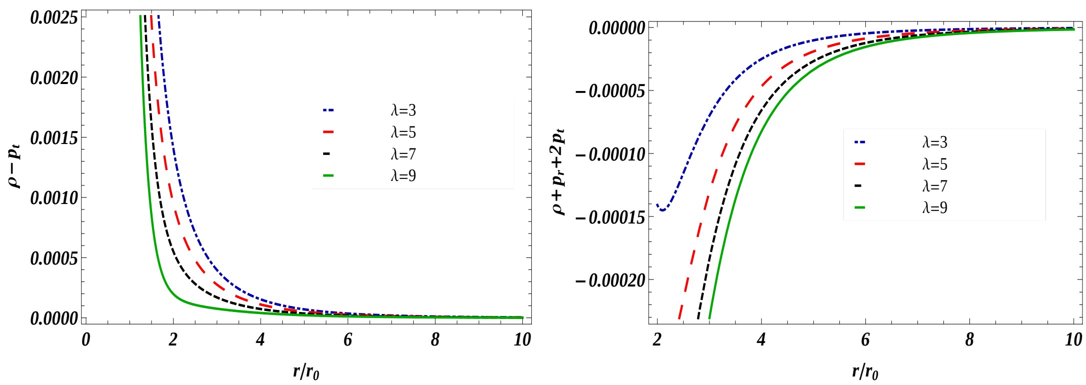

Fig. 13: Shows the development of $\rho - p_t$ and $\rho + p_r + 2p_t$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

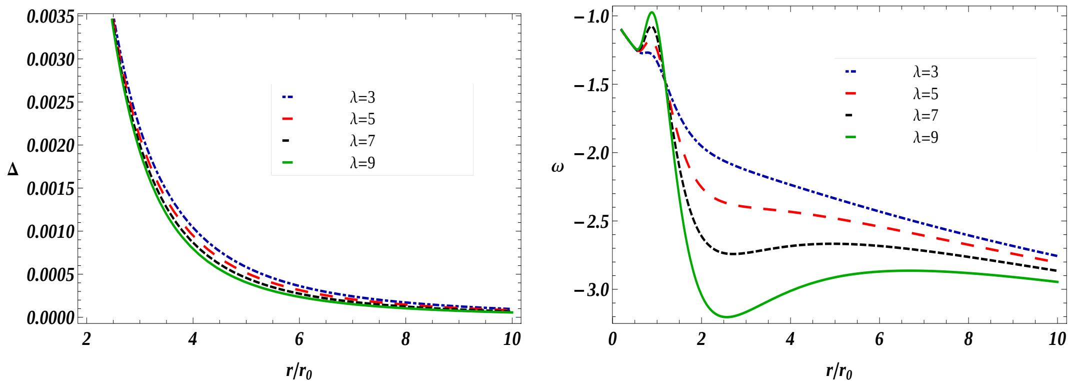

Fig. 14: Shows the development of Δ and ω with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

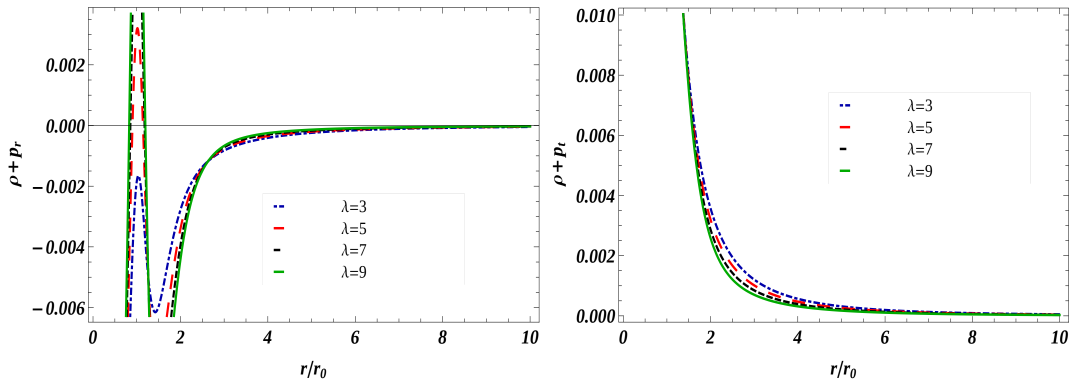

Fig. 15: Shows the development of $\rho + p_r$ and $\rho + p_t$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

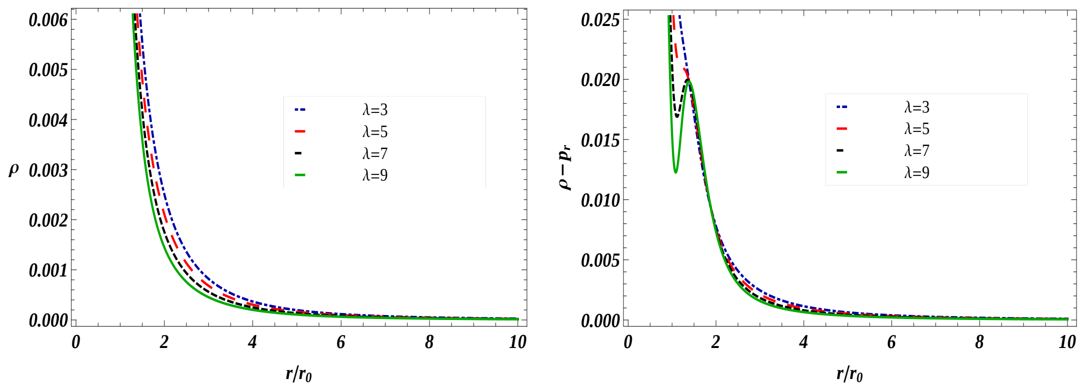

Fig. 16: Shows the development of ρ and $\rho - p_r$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

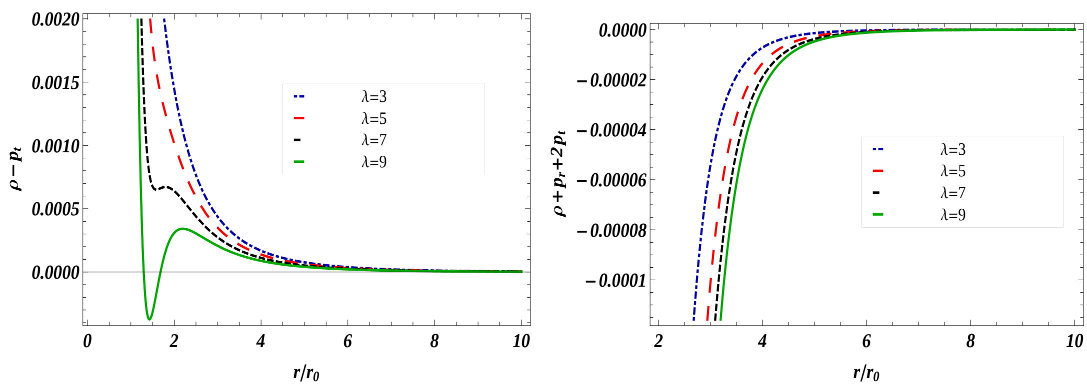

Fig. 17: Shows the development of $\rho - p_t$ and $\rho + p_r + 2p_t$ with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

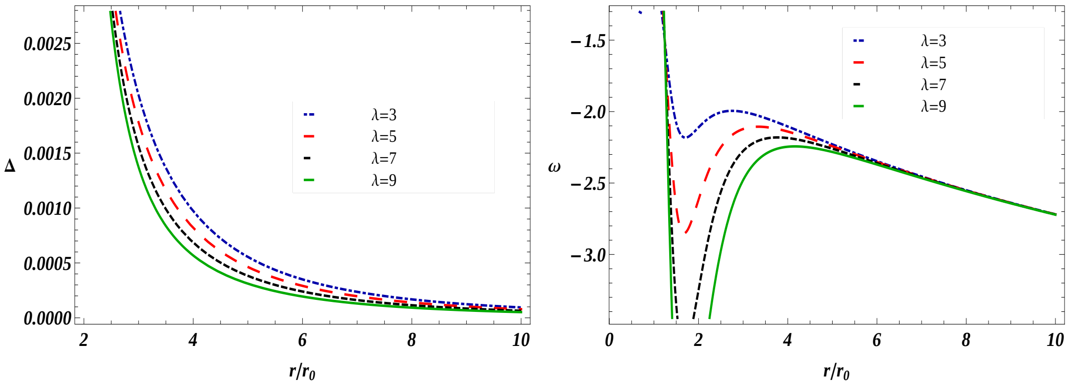

Fig. 18: Shows the development of Δ and ω with $n = 5$, $a_1 = 2$, $r_0 = 0.9$ and $\gamma = 0.5$.

5. NEC and WEC for $b(r) = r_0 \log(\frac{r}{r_0} + 1)$

In this section, energy conditions for a specific shape function (Godani, 2018., Smantha, 2018), which is defined as:

$$b(r) = r_0 \log(\frac{r}{r_0} + 1) \tag{36}$$

Now, by using Equation (36) in both exponential gravity models, which are given in Equation (27) and in Equation (31) we can check the behavior of energy conditions. As first attempt, we calculated the generic field equations for both models by Equation (27) and Equation (31) in Equations (15-17). The energy density and pressure components for the first model by Equation (27) are calculated as:

$$\rho = \frac{b'(r)\left(1-\alpha e^{-\frac{2b'(r)}{\gamma r^2}}\right)}{(\lambda+8\pi)r^2}, \tag{37}$$

$$\begin{aligned} p_r &= \frac{e^{-\frac{2b'(r)}{\gamma r^2}}}{\gamma^2(\lambda+8\pi)r^7}\Bigg(b(r)\left(\alpha\left(-16b'(r)^2+r^2\left(\gamma r^2\left(2b^{(3)}(r)+\gamma\right)-9\gamma r b''(r)-4b''(r)^2\right)\right.\right. \\ &+ \left.\left. 2rb'(r)\left(8b''(r)+7\gamma r\right)\right)-\gamma^2 r^4 e^{\frac{2b'(r)}{\gamma r^2}}\right)+\alpha r\left(-2\gamma r^4 b^{(3)}(r)+4r^2 b''(r)\left(b''(r)+2\gamma r\right)\right. \\ &= \left. 2\left(\gamma r^2-8\right)b'(r)^2+rb'(r)\left(\left(\gamma r^2-16\right)b''(r)-12\gamma r\right)\right)\Bigg), \end{aligned} \tag{38}$$

$$p_t = \frac{r^2\left(b(r)-rb'(r)\right)+\frac{\alpha e^{-\frac{2b'(r)}{\gamma r^2}}\left(r\left(\left(\gamma r^2+8\right)b'(r)-4rb''(r)\right)-b(r)\left(r\left(\gamma r-4b''(r)\right)+8b'(r)\right)\right)}{\gamma}}{2(\lambda+8\pi)r^5}. \tag{39}$$

The energy density and pressure components for the second model by Equation (31) are calculated as:

$$\rho = \frac{b'(r)\left(1-\alpha\text{sech}^2\left(\frac{2b'(r)}{\gamma r^2}\right)\right)}{(\lambda+8\pi)r^2}, \tag{40}$$

$$\begin{aligned} p_r &= \frac{1}{2\gamma^2(\lambda+8\pi)r^7}\Bigg(-48\alpha(r-b(r))\left(rb''(r)-2b'(r)\right)^2\text{sech}^4\left(\frac{2b'(r)}{\gamma r^2}\right) \\ &+ 2\alpha\left(b(r)\left(r\left(4b''(r)+\gamma r\right)-8b'(r)\right)\left(r\left(\gamma r-4b''(r)\right)+8b'(r)\right)+16r\right. \\ &\times \left.\left(rb''(r)-2b'(r)\right)^2\right)\text{sech}^2\left(\frac{2b'(r)}{\gamma r^2}\right)+4\alpha\gamma r^2\left(-2rb'(r)^2+r\left(2r(b(r)-r)\right.\right. \\ &\times \left. b^{(3)}(r)+(8r-9b(r))b''(r)\right)+b'(r)\left(r\left(rb''(r)-12\right)+14b(r)\right)\Big) \\ &\times \tanh\left(\frac{2b'(r)}{\gamma r^2}\right)\text{sech}^2\left(\frac{2b'(r)}{\gamma r^2}\right))-2\gamma^2r^4b(r)\Bigg), \end{aligned} \tag{41}$$

$$\begin{aligned} p_t &= \frac{1}{2\gamma(\lambda+8\pi)r^5}\Bigg(\alpha\gamma r^2\left(rb'(r)-b(r)\right)\text{sech}^2\left(\frac{2b'(r)}{\gamma r^2}\right)+\gamma r^2\left(b(r)-rb'(r)\right) \\ &- 8\alpha(r-b(r))\left(rb''(r)-2b'(r)\right)\tanh\left(\frac{2b'(r)}{\gamma r^2}\right)\text{sech}^2\left(\frac{2b'(r)}{\gamma r^2}\right)\Bigg). \end{aligned} \tag{42}$$

Since we know that the existence of exotic matter in a wormhole is correlated to the violation of NEC. In response to the above mentioned energy conditions, we observe that for both models energy density is positive and decreasing (Figure 13) and (Figure 16)for the specific shape function i.e. $b(r) = r_0\log(\frac{r}{r_0}+1)$. Development of $\rho+p_r$ and $\rho+p_t$ (NEC and WEC) can be seen in (Figure 11) and (Figure 15) respectively. For both models, NEC is violated because of the negative behavior of $\rho+p_r$, which is evidence of the presence of exotic matter. All the energy conditions are presented graphically for $b(r) = r_0\log(\frac{r}{r_0}+1)$ specific model in (Figure 11-18) for both considered model of $f(R,T)$ gravity.

6. Conclusion

Scientists have always been concerned about the construction and occurrence of wormhole solutions in GR. The existence of exotic matter is one of the significant settings for producing wormholes because it violates NEC. But this is not the compulsory condition for the wormhole presence as long as it is cosmologically acceptable. Wormhole construction is a fascinating topic in terms of modified theories of gravity. In this paper, we have discussed different cases in $f(R,T)$ gravity and shown that the existence of exotic matter is a necessary but not critical condition for wormhole solutions existence. The essential features of the current study are itemized below:

- Firstly, we considered inhomogeneous spacetime in the $f(R,T)$ theory of gravity. Wormhole solutions are found using two captivating $f(R)$ models i.e. exponential gravity model ($f_1(R) = R-\alpha\gamma(1-e^{-\frac{R}{\gamma}})$) and Tsujikawa model ($f_1(R) = R-\alpha\gamma tanh(\frac{R}{\gamma})$). The energy conditions are studied and their geometric nature is examined. NEC, WEC and DEC are satisfied everywhere for both models (Figures 3-5, and Figures 7-9). This shows the presence of solutions of wormholes without the existence of exotic matter as NEC is satisfied everywhere. The geometric nature of the wormhole is found to be repulsive and is full of non-phantom fluid (Figures 7, 10). Hence, the obtained results show the existence of solutions of wormholes in the presence of non-exotic matter.

- Moving forward, NEC an WEC for a specific shape function i.e. $b(r) = r_0\log(\frac{r}{r_0}+1)$ using both exponential gravity model and Tsujikawa model have been explored. We observe that for the given models, energy conditions are provided in (Figures 11-18), and NEC is violated because of the negative behavior of $\rho+p_r$ which is the evidence of the existence of exotic matter.

In a nutshell, the present study gives a detailed discussion on the occurrence of solutions of wormholes in $f(R,T)$ gravity using specific models. In particular, we may conclude from this work that traversable wormholes are conceivable with or without the presence of exotic matter.

Acknowledgements G. Mustafa is very thankful to Prof. Gao Xianlong from the Department of Physics, Zhejiang Normal University, for his kind support and help during this research. Further, G. Mustafa acknowledges Grant No. ZC304022919 to support his Postdoctoral Fellowship at Zhejiang Normal University.

References

Agnese A. G., and Camera, M. La. (2011). Wormholes in Brans-Dicke Theory of Gravitation, *Phys. Rev.* **D51**(1995).

Alvarenga, F. G.(2013). Dynamics of scalar perturbations in $f(R,T$ gravity, Phys. Rev. **D87**, 103526.

Bronnikov, K.A.(1973). Scalar tensor theory and scalar charge, Acta Phys. Pol. **B4**, 251.

Bronnikov, K. A, and Galiakhmetov, A. M. (2015). Wormholes without Exotic Matter in Einstein-Cartan Theory, Grav. Cosmol. **21**, 283.

Bronnikov, K. A, and Galiakhmetov, A. M. (2016). Wormholes and Blackholes without Phantom Fields in Einstein-Cartan Theory, Phys. Rev. **D94**, 124006.

Baffou, E. H.(2015). Cosmological viable $f(R,T)$ dark energy model: dynamics and stability, Astrophys. Space Sci. **356**, 173-180.

Clement, G.(2004). A class of wormhole solutions to higher-dimensional general relativity, Gen. Relativ. Gravit. 16(1984)131.

Carrol, S. M., Duvvuri, V., Trodden, M., and Turner, M. S., (2008). Is Cosmic Speed-up due to New Gravitational Physics? Phys. Rev. **D70**, 043528.

Cognola, G.(2008). Class of viable modified $f(R)$ gravities describing inflation and the onset of accelerated expansion, Phys. Rev. **D77**, 046009.

Dzhunushaliev, V. and Singleton, D. (1999). Wormholes and Flux Tubes in 5D Kaluza-Klein Theory, Phys. Rev. **D59**, 064018.

Deffayet, D., Dvali, G., and Gabadadze, G. (2002). Accelerated Universe from Gravity Leaking to Extra Dimensions, Phys. Rev. **D65**, 044023.

DeFelice, A. and Tsujikawa, S.(2010). $f(R)$ theories, Living Rev. Rel. **13**, 3.

Einstein, A. and Rosen, N.(1935). The particle problem in the General theory of relativity, Phys. Rev. **48**- 73.

Ellis, H.G., (1973). Ether flow through a drainhole: A particle model in general relativity, J. Math. Phys. **14**, 104.

Eiroa, E.F., Richarte. M.G., and Simeone, C.(1973). Thin-shell wormholes in Brans-Dicke gravity, Phys. Lett. A **373**, (2008)1-4.

Eiroa E. F., and Aguirre G. F.(2012). Thin-shell Wormholes with a Generalized Chaplygin Gas in Einstein-Born-Infeld Theory, Eur. Phys. J. **C72**, 2240.

Elizalde, E.(2011). Non-singular exponential gravity: a simple thoery for early and late-time accelerated expansion, Phys. Rev. **D83**, 086006.

Flamm L.(1916). Comments on Einstein's Theory of Gravity, Phys. Z **17**, 448.

Falco et al., (2020). General relativistic Poynting-Robertson effect to diagnose wormholes existence: Static and spherically symmetric case, Phys. Rev. D **101**, 104037.

Godani, N., and Samanta, G. C.(2019). Non Violation of Energy Conditions in Wormhole Modeling, Mod. Phys. Lett. **A34**, 1950226.

Godani, N., and Samanta, G. C.(2018). Traversable wormholes and energy conditions with two different shape functions in $f(R)$ gravity, Int. J. Mod. Phys. **D28**, 19950039.

Golchina, H., Mehdizadeh, M. R., (2019). Quasi-cosmological traversable wormholes in $f(R)$ gravity, Eur. Phys. J. C 79:777.

Harko, T, Lobo, F. S.N., Nojiri, S., and Odintsov, S. D. (2011). $f(R,T)$ Gravity, *Phys. Rev.* **D84**, 024020.

Houndjo, M. J. S. (2012). Reconstruction of $f(R,T)$ gravity describing matter dominated and accel-

erated phases, Int. J. Mod. Phys. **D21**, 1250003.
Houndjo, M. J. S.(2012). Reconstruction of $f(R,T)$ Gravity Describing Matter Dominated and Accelerated Phases, Int. J. Mod. Phys. D21, 1250003.
Jamil et al.,(2012). Reconstruction of some comological models in $f(R,T)$ cosmology, Eur. Phys. J. **C72**, 1999.
Lobo, F. S. N. and Oliveira, M. A. (2010). General Class of Vacuum Brans-Dicke Wormholes, Phys. Rev. **D81**, 067501.
Morris, M.S., Thorne, K.S., (1988). Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, Am. J. Phys. **56**, 395.
Mazharimousavi, S. H., Halilsoy, M. and Amirabi, Z. (2010). Stability of Thin-shell Wormholes Supported by Normal Matter in Einstein-Maxwell-Gauss-Bonnet Gravity, Phys. Rev. **D81**, 104002.
Mazharimousavi, S. H., Halilsoy, M. and Amirabi, Z. (2011). Higher-Dimensional Thin-Shell Wormholes in Einstein-Yang-Mills-Gauss-Bonnet Gravity, Classical Quant. Grav. **28**, 025004.
Mehdizadeh, M. R., Zangeneh, M. K., Lobo, F. S. N. (2015). Einstein-Gauss-Bonnet Traversable Wormholes Satisfying the Weak Energy Condition, Phys. Rev. **D91**, 084004.
Mehdizadeh, M. R. and Ziaie, A. H. (2017). Einstein-Cartan Wormhole Solutions, Phys. Rev. **D95**, 064049.
Moraes, P. H. R. S, Correa, R. A. C., and Lobato, R. V. (2017). Analytical General Solutions for Static Wormholes in $f(R,T)$ Gravity, JCAP **07**, 029.
Nandi, K. K., Islam, A., and Evans J., (1997). Brans Wormholes, Phys. Rev. **D55**, 2497.
Nojiri, S., and Odintsov, S. D.,(2003). Modified Gravity with Negative and Positive Powers of Curvature: Unification of Inflation and Cosmic Acceleration, Phys. Rev. **D68**, 123512.
Richarte. M.G., and Simeone, C. (2007). Wormholes in Einstein-Born-Infeld Theory, Phys. Rev. D **76**, 087502.
Sharif, M., and Mumtaz,S. (2016). Influence of nonlinear electrodynamics on stability of thin-shell wormholes, Astrophys. Space Sci. **361**, 218.
$\ddot{O}$**vgun, A.(2018).** Light deflection by Damour-Solodukhin wormholes and Gauss-Bonnet theorem, Phys. Rev. D **98**, 044033.
Sushkov, S. V., and Kozyrev, S. M.(2011). Composite Vacuum Brans-Dicke Wormholes, Phys. Rev. **D84**, 124026.
Shaikh R. and Kar S. (2016). Wormholes, the Weak Energy Condition, and Scalar-tensor Gravity, Phys. Rev. **D94**, 024011.
Singh, C. P., and Singh V.(2014). Reconstruction of modified $f(R,T)$ gravity with perfect fluid cosmological models, Gen. Relativ. Gravit. **46**, 1696.
Shabani H., and Farhoudi M.(2013). $f(R,T$ cosmological models in phase space, Phys. Rev. **D88**, 044048.
Shabani H., and Farhoudi M.(2014). Cosmological and solar system consequences of $f(R,T$ gravity models, t Phys. Rev. **D90**, 044031.
Santos, A. F.(2013). Gödel solution in $f(R,T)$ gravity, Mod. Phys. Lett. **A28**, 1350141.
Shamir, M. F, Mustafa, G. and Fazal. A. (2021). Non-commutative Wormhole Solutions in Exponential Gravity with Matter Coupling, New Astron. **83**, 101459.
Smantha, G. C, Godani, N. and Bamba, K. (2018). Traversable wormholes with exponential shape function in modified gravity and in general relativity: A comperative study, arXiv:1811.06834.
Tsujikawa, S.(2008). Observational signatures of $f(R)$ dark energy models that satisfy cosmological and local gravity constraints, Phys. Rev. **D77**, 023507.
Zangeneh, M. K, Lobo, F. S. N. and Dehghani, M. H. (2015). Traversable Wormholes Satisfying the Weak Energy Condition in Third-order Lovelock Gravity Phys. Rev. **D92**, 124049.

Submitted: 25/01/2021
Revised: 22/05/2022
Accepted: 25/07/2022
DOI: 10.48129/kjs.12005