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Root extraction by Niẓām al-Dīn al-Nīsābūrī (d. ca. 1330/730AH)

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Abstract

In this article, we discuss root extraction in al-Ḥasan al- Nīsābūrī (d. Ca. 1330 AD), based on six copies of his manuscript titled<<*al-Risāla al-Shamsiyya fil-Ḥisāb*>>. Most of the studies we have consulted present al-Nīsābūrī as a religious scholar, an exegete of the Qur'ān, and linguist, but they do not address his scientific production especially the mathematical one. Hence, the benefit of this research is to initiate a new avenue in the study of the transfer of the mathematical results and concepts between different scientific foci in the Arab-Islamic civilization. This research addresses the absence of studies dealing with some important aspects of overall mathematics produced in central Asia. The importance of this study lies in the fact that it represents a link between what was produced in this field in the 13th century, by al-Ṭūsī(d. 1274) in his work <<*Jawāmi al-Hisāb bi al-Takht wa al-Turāb*>> and in the 15th century, by al-Kāshī(d. 1429) in his treatise <<*Miftāḥ al-ḥisāb*>>, in terms of tools and techniques.

Keywords: Al-Ḥisāb, Al-Shamsiyya fil-Ḥisāb, Al- Nīsābūrī, Square Root, Root extraction.

1. Introduction.

Al-Nīsābūrī era generally falls in the period between the second half of the thirteenth century and the first half of the fourteenth century, which coincided with the Mongol rule of Persia region and the consequent changes at all geopolitical, scientific, and economic levels. This century is considered one of the pre-eminent centuries in the history of the Arab-Islamic civilization, which was full of political, economic, cultural, and scientific events. This scientific movement, within its quantitative component, emerges in Rosenfeld's and Ihsanoğlu's book" Muslim Astronomers and Mathematicians" (Rosenfeld, B. A. & E. Ihsanoğlu. (2003)) in which they count more than 270 Muslim scholars (astronomers and mathematicians) who lived, partly or entirely in the 13th and 14th centuries. If we further consider what recent research and studies after 1900 have shown, this number will be much higher (Djebbar A. (2004), p. 159).

In the beginning, we will briefly talk about the different political, cultural, and scientific events that took place in the region at the time, about the phenomenon of the Mongol and their seizing of power in the Islamic countries.

Then, we will present al-Nīsābūri's biography based on the scarce information available within the literature. Therein, we shed light on the scholar's scientific and religious education, as well as his works in Astronomy, Mathematics, and other fields of Knowledge.

After that, we will introduce al-Nīsābūr's famous book in mathematics *al-Risāla al-Shamsiyya fil-Ḥisāb* (Treatise on Arithmetic) and its main contents and sections. We will study the first part of the second section of this book which addresses the extraction of square and n^{th} roots and methods of their approximation and compare that with the works of Naṣīr al-Dīn al-Ṭūsī (d. 1274) and Ibn al-Bannāʾ al-Marrākushī (d. 1321) from the Islamic Maghreb. Subsequently, we will see the extent of this work, which lasted until after the fourteenth century as demonstrated in al-Kāshī's works (d. 1429).

2. Mongol Phenomenon.

The Mongol entry in the Islamic countries and the emergence of their great state, which included Iran (Iran now), and China, Mesopotamia, and Asia Minor are considered some of the most important historical events that characterized the thirteenth and fourteenth centuries.

The events related to the Mongol phenomenon could be divided into three periods: The first period, which has no direct relation with our topic, coincides with the beginning of the Mongol entry in the Khwarazmian state in 1219 under the leadership of the Mongolian leader Genghis Khān (d. 1227)(Al-Sayad, F. A. (1980), p. 308).

The second campaign began in early 1253, and it was led by Hülegü Khān (1217 - 1265), headed to Azerbaijan where he chose Marāgha - in the north of the province - as the capital of his kingdom. Hülegü's conquests extended to Asia Minor, the lands of the Bulgarians, and Eastern Europe. After that, a new state in Iran known as The Īlkhānids State was established.

By the end of the thirteenth century, the Mongols converted to Islam in the reign of Ghāzān Khān (1295-1304) (Brockelmann, C. (1984), p. 391), who broke away from the Grand Khān in Beijing; he encouraged economic development and reconstruction in the abandoned areas due of wars; established two schools of the Shafi'i and Hanafi Wqāfs and observatory on the outskirts of Tabriz (Ashtiani's, A. I. (1989), p. 393).

In the month of Jumada I 657 AH (April-May 1259) (Haji, Khalifa. (1941), p. 967), Hülegü Khān entrusted the construction of an astronomical observatory to Scholar Naṣīr al-Dīn al-Ṭūsī in Marāgha, about 130 *km* from the city of Tabriz. Al-Ṭūsī, the director of the observatory, was surrounded by a group of leading astronomers, such as Muʾayyad al-Dīn al-ʿUrḍī (d. 1266), Muḥyi al-Dīn al-Maghribī (d. 1283), Qutb al-Dīn Al-Shirāzī (d. 1311), in addition to a Chinese astronomer named Fu Mengchi *(*Fu Muzhai). Those scholars established new astronomical Zijs (zij: Persian word, Romanised: zīj, astronomical catalogs which used for astronomical calculations), such as one known as *Zīj-i Īlkhānī* (A. Sayili. (1998), pp. 189-223; M. Régis & Roshdi, R. (1997), pp. 13-14).

Marāgha observatory attracted many students from the Islamic world. This observatory continued to operate and remained active until 715AH/ 1316AD (i.e. the reign of 7 Mongol rulers after Hülegü Khān - until the end of the reign of Öljeitü) which was the date of the death of its last known director Aṣil-al- Dīn (the son of Naṣīr al-Dīn al-Ṭūsī (A. Sayili. (1998), p. 213).

3. Life and Works of Al- Nīsābūrī.

Niẓām al-Dīn al-Ḥasan ibn Muḥammad ibn al-Ḥusayn al-Qummi Al-Nīsābūrī, also known as Niẓām al-A'raj al-Nīsābūrī (Haji Khalifa. (1941), p. 1195; Qurbānī, Abu al-Qasim. (1375AH), p. 507), is from a Shiite family whose roots trace back to the city of Qum (is a city in Iran, located 125 *km* south of Tehrān). He was born in Nishapur, the most important city in Khorasan province (modern-day Iran).

Nishapur had a rich history of science from the early times of Islam until the 11th century, but the city was invaded by the Mongols in 1221 and do not develop as it once had. Ibn Baṭūṭah visited Nishapur, or "Little Damascus" as he named it, in the 14th century; he visited its landmarks and schools including four schools that had many students and described them as good schools (Ibn Baṭūṭah, p. 353). Biographers estimate that al-Ḥasan al-Nīsābūri's birth coincides with al-Ṭūsī (d. 1274) death.

Morrison mentions that al-Nīsābūrī was born around 1270 (R. Morrison. (2007), p. 7). His life was generally between the end of the 13th century and the beginning of the 14th century (A. Sayili. (1998), p. 189).

We do not have much information about al-Nīsābūrī's childhood and early education until 1300; except that al-Nīsābūrī learned the Holy Qur'ān when he was a child (Al-Nīsābūrī. (1996), pp. 5-6), and that he learned the Arabic language and its grammar in addition to the Persian language; <<*His high reputation in merits, erudition, capacity to dig into science and excellence of mind were famous; he was also one of the most significant memorizers (Hafiz Al Qur'ān) and exegetes who reached the degree of Jalāl-al-Dīn al-Dawwānī and* Ibn Ḥajar al-ʿAsqalānī *and their counterparts*>> (Al-Mūsawī M. B. (1991), p. 96). He studied Arabic grammar (an-naḥw) from most important books such as Sībawayh (d. 796), and "*Shāfiyya*" of ibn al-Ḥājib (d. 1249).

Al-Nīsābūrī was a renowned astronomer, an exegete of Qur'ān, and a fine expert in philosophy and literature. His studies in philosophy were based on the works of Ibn Sīnā (d. 1037), and Abū Ḥāmid al-Ghazālī (d. 1111) and F̱akhr al-Dīn al-Rāzī (d. 1209) in *ʿIlm al-Kalām* (Islamic theology) and *Fiqh* (Islamic jurisprudence).

Biographical and history books and sources say little about al-Nīsābūrī teachers; his only known teacher is Quṭb al-Dīn al-Shirāzī (Qurbānī, Abu al-Qasim, (1375AH), p. 507), one of Marāgha observatory scholars with whom he worked in Tabriz during the reign of Ghazan Kahn (1295-1304) and Öljeitü (1304-1317) (Rosenfeld B. A. & E. Ihsanoğlu. (2003), pp. 238-239) benefiting from his works: *"Nihāyat al-idrāk fī dirāyat al-aflāk"* (The Limit of Accomplishment concerning Knowledge of the Heavens), "*Al-Tuḥfa al-šāhiya fi'l-hayʾa*"(an Arabic book on astronomy, having four chapters) and "*fa'altu fa-lā talum*"(I Have Done [what I did] But Do Not Blame [Me for It], on Astronomy).

We know that the astronomical treatise "*Zīj al-`Ala'i*" was corrected by al-Nīsābūrī's*,* disciples after his death (Haji Khalifa. (1941), p. 266); this proves that al-Nīsābūrī had disciples and students; we, though, do not have much information about this matter more than what we have mentioned above.

Al-Nīsābūrī arrived in Azerbaijan in 1304(R. Morrison. (2007), p. 37), then he went to Tabriz –one of the most important cities in Azerbaijan and Ilkhanids capital– around 1305 where he completed his book "*Sḥarh Taḥrīr al-Majisti*" (Commentary on the recession of the Almagest) on Mars 4*th*, 1305(Qurbānī. Abu al-Qasim., (1375AH), p. 507), and included in it the results of his works. "*Taḥrīr al-Majisti*" (Recession of the Almagest) is one of the most important al-Ṭūsī astronomical texts in which he provided commentary on astronomy as presented by Greek astronomer Ptolemy(ca. 2nd c. CE) in his famous book "*Almagest*".

Al-Nīsābūrī studied al-Ṭūsī's works and wrote commentaries on them; indeed, al-Ṭūsī was a role model to al-Nīsābūrī and a reason for his intellectual development (R. Morrison. (2007), p. 37). In the period between 1308 and 1309(R. Morrison. (2007), pp. 148-149), al-Nīsābūrī completed his second most significant work in Persian entitled "*Kashf al-ḥaqāʾiq-i Zīj*-i *Īlkhānī* " (Uncovering the truths of the Ilkhanid Zij) which was an important guide to astronomy; it is a commentary on al-Ṭūsī astronomical handbook "*Zīj*-i *Īlkhānī* " as well.

"*Tawḍīḥ al-Tadhkirah*" (Elucidation of the Tadhkira) is al-Nīsābūrī's third important astronomical work. These texts by al Nīsābūrī were taught in schools (R. Morrison. (2007), p. 17) such as Ulūgh Beg (d. 1449) school in Samarkand; Al-Kāshī (d.1429), moreover, mentions that his father told him that he had studied al-Ṭūsī's "*al-tadhkira fī ʿilm al-hayʾa*" (Memento on astronomy) and its elucidation by al-Nīsābūrī "*Tawḍīḥ al-Tadhkirah*" in Samarkand (R. Morrison., (2007), p. 17).

Al-Nīsābūrī wrote several treatises in astronomy, some of which are: "Sharḥ *Sī faṣl*" (H. Suter. (1986), p. 161) a commentary on al-Ṭūsī's work (Thirty chapters on the science of the calendar), "*Sharḥ bīst bāb dar asṭurlāb*" also a commentary on (Twenty Chapters Dealing with the Uses of the Astrolabe) of al-Ṭūsī's "*al-ʻamal bi-al-rubʻ al-muqanṭar*" which is as well a commentary on al-Ṭūsī's work "*Rubʻ al-muqanṭar*", "*Risāla fī maʻrifat samt al-qiblah*" (On finding the direction of the Qibla) and "*al-Baṣair fi mukhtaṣar Tanqīḥ al-Manāẓir*" (Enlightening of the topics in al-Manazer).

Al-Nīsābūrī composed several books in linguistics and religious sciences mainly a commentary of Qur'ān entitled "*Gharāʾib al-Qurʾān wa-raghāʾib al-furqān*" (Al-Ṭabarī. (1980)) (The curiosities of the Qur'ān and the desiderata of the demonstration) which is printed in the margin of ibn Jarīr al-Ṭabarī's (d. 923) book *"Jāmiʿal-bayān*" in which he based on F̱akhr al-Dīn al-Rāzī's book "*Al-Tafsīr al-Kabīr*" (The great commentary) and included what he had found in "*Al-Kashshāf*" commentary by al-Zamakhsharī (d. 1144) (Al-Nīsābūrī. (1996), p. 6) and completed it in the month of Safar 730AH/ November-December 1329 (Morrison. (2007), p. 152). Similar to this work, al-Nīsābūrī wrote several texts such as *Sharḥ al-asmā' al- ḥusnā* (Commentary on the most beautiful Names) which was completed in Shawwal 710 AH (Morrison. (2007), p. 151), *Risāla fī bayān farā'iḍ al-ṣalāh* (Epistle of the Exposition of the Precepts of Prayer), *Tafsīr al-Fātihāt* (a commentary on *al- Fātihā*), "*Lubāb al-tā'wīl fī tafsīr al-Qur'ān*" in one volume (al-Baghdādī I. B. (1992), p. 283), "*Awqāf al-Qurʾān*" in the science of pausing «*al-Wqāf*» and resuming «*al-Ibtidā*'» in Qurʾān, "*Sharḥ Shāfiyyah*" in Arabic grammar by ibn al-Ḥājib, *Sharḥ Miftāḥ al-ʿulūm* by al-Sakkākī (d. 1229), and *Risāla fī maʿrifāt al-jumāl nākirat am lā'*.

4. Al-Nīsābūrī's Mathematical Production.

This work is known as «*al-Risāla al-Shamsiyya fil-Ḥisāb*» (*Epistle on Arithmetic*) [35, Ms. 3149 fol. 2a]. Al-Nīsābūrī dedicated a copy of "*al-Shamsiyya*" to Shams al-Din ʿAbd al-Laṭīf, the son of Rashīd al-Din al-Hamadāni (the minister of Mongol - Ḡāzān and Öljeitü- historian, author of the book " *Jāmeʿ al-tawārik̲* (The Compendium of chronicles)), whose name (*al-Shamsiyya*) maybe has been extracted from his name. (Al-Nīsābūrī. Ms. 3152 fol. 3b; Ms. 2725 fol. 2b). But, we do not have much information about the date of composition of this famous work (Youschkevitch A. P. (1976), p. 168).

According to the information in our hands, we have counted more than 156 copies of *al-Shamsiyya fil-Ḥisāb* in many places in the world among which are: Iran 69 copies (catalog of Iranian Manuscript (Dana), Volume I, Tehrān: Library), Turkey (Istanbul) 15 copies (King D. A. (1986), pp. 896-897); (H. Suter. (1900), p. 161; Rosenfeld B. A. & E. Ihsanoğlu. (2003), p. 238), Uzbekistan12 copies(Tashkent 11 copies, Bukhara a single copy), Egypt 7copies (Cairo 4 copies, Institute of Arabic manuscripts 3 copies), Saudi Arabia 5 copies (Umm al-Qurā University in Mecca 2 copies, King Abdul-Aziz University 3 copies, King Faisal Center for Research and Islamic Studies one copy), UAE 3 copies (Juma Al Majid Centre for Culture and Heritage, UAE), and a single copy in Holland (Al-Nīsābūrī. Ms. Or., n° 204).

Several commentaries have been written on al-Risāla among them we mention a commentary by 'Abd al-'Alī ibn Muḥammad ibn Ḥusayn Birjandī (d.1527) in 924AH /1518AD called Sharḥ al-Shamsiyya (He says at the beginning <<*... Arithmetic is one of the noblest sciences... "Al-risāla-al-shamsiyya" was among what has been written in the "Risālat Suniyya" (Sunni treatises) by Niẓām al-Milla wa- al-Dīn al Ḥasan al-Nīsābūrī ... this treatise has got the great benefit that includes what is needed as types of arithmetic and rules; therefore, it's as famous everywhere as the sun at high noon...*>>.('Abd al-'Alī Bīrjandī, Ms. 00116, fol. 2a)

5. The Mathematical Content of *Al- Shamsiyya.*

The treatise is divided into an introduction and two sections. The introduction includes the definition of the science of calculations, its bases and its importance, definition and, and types of numbers, then the nine figures and their places.

The first section is devoted to the principles of arithmetic and operations on numbers: duplication and mediation, addition and subtraction, multiplication and division, calculations with fractions, their denominators and operations on them and replacing a denominator with another denominator. He second section deals with the definition of different powers, extracting square roots or roots of higher-order, using the letters of the Arabic alphabet to denote numerals, using the Sexagesimal number system, (*ḥisāb al-jummal*), extracting the square root in the Sexagesimal system, operations on the monomials (defining the monomials,x, x^2, x^3,… and $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \ldots$ and giving rules for multiplication and division). In addition to this a part in geometry includes some definitions such as line and point, etc., area of familiar geometric shapes such as triangle, square, etc. and calculation of some famous volumes. Then an appendix contains two chapters, one on *al-jābr wa'l-muqābala* (Algebra), and the other about *ḥisāb al-mīzān* (Checking of Calculations)*, mīzān* (= balance) and Regula-Falsi Method (double false position Method).

6. Root extraction.

Calculating roots occupied a large portion of the Arabic works of arithmetic written both in the Orient and in the Occident. Since the ninth century, many books of arithmetic (books of *al-ḥisāb*) dealt with extracting square and cube root, and sometimes more broadly, the n^{th} root of integers. In his today missing book *al- ḥisāb al-ḥindī* (the Hindu Calculation), al-Khwārizm*ī* included a method to extract approximate square roots (the content of this book is known due to Latin copies. See: (A. Allard. (1992)).

Aḥmad ibn Ibrāhīm Abū al-Ḥasan al-Uqlīdisī (10th century), as well had been interested in this issue in his book "*Al-Fuṣūl fil-ḥisāb al-Hindī*" (The Book of Chapters on Hindu Arithmetic) (Al-Uqlīdisī. (1984), pp. 490-503) in which he included rules for approximating square and cube roots. Ibn Tāhir al-Baghdādī (d.1037) also dealt with extracting square and cube roots in his treatise *al-Takmila fi'l-ḥisāb* (the Completion of Arithmetic) (Al-Baghdādī. (1985), pp. 72-80; 84-94). Luckey mentioned that al-Nasawī (11th century), had used what would be later known as Ruffini–Hörner method (19th century), to extract cube roots for the first time (Al-Kāshī. (1969), p. 278).

Al-Nasawī's text on the cube root was translated into German by Luckey (B.G. Johansson. (2011), p. 346). The methods used in cube root extraction as described in these works have obvious common features with the methods used in China: such as, Jia Xian's (11th century) algorithm.

Generally speaking, after the mathematicians have used the Numerical Triangle or Pascal's triangle and the Binomial formula by the end of 10th century, besides the development of Algebra at that time, they no longer faced difficulties in formulating algorithms to extract roots of higher-order. Al-Bīrūnī (d. 1048) and Al-Khayyām (d. 1131) made such attempts by the end of the 11th century, and the beginning of the 12th century, but all those attempts were regrettably lost (Youschkevitch A. P. (1976), pp. 76-80).

Al-Samaw'al al-Maghribī's (d. 1175) contribution appears in his book *al-Qiwāmi fi l-ḥisāb al-hindī* (Treatise on Indian Arithmetic) in which he uses Ruffini–Hörner's method in extracting the 5^{th} root of a sexagesimal positive integer number, and he formulates a clear perception of its approximating (Roshdi R. (1989), pp. 115-127). This activity would continue until the end of the 13th century, in al-Ṭūsī's treatise *Jawāmi' al-ḥisāb bi al-takht wa al-turāb* (Arithmetic of the board and the dust) in which he explains a method to extract the root of higher-order as well as the Binomial Theorem (Al-Uqlīdisī. (1984), p. 508).

7. Root extraction in al-Nīsābūrī's work.

Al- Nīsābūrī defines the square of a positive integer as a number multiplied by itself; there are two types of it: perfect square (spoken) *munṭaq*, imperfect square (dumb) *aṣamm*. Which raises the issue of root extraction algorithm and its approximation?

According to al-Nīsābūrī, the square root extraction algorithm of a positive integer is based on writing the number into the decimal positional system in the famous identity,

$$(a+b)^2 - b^2 = 2ab + a^2 \qquad (5.1)$$

7.1. Algorithm of the square root extraction of a positive integer N.

- Find the biggest simple number (*'adad mufrad)* d_p(a number with only one non-zero digit, like 5, 300, or 6,000)

$d_p = c_p.10^p \,/\, 1 \le c_p \le 9$ such that: ${d_p}^2 \le N$

If ${d_p}^2 = N$ then, $\sqrt{N} = d_p$.

- Else (${d_p}^2 < N$)

Find the biggest simple number d_{p-1} such that, $\left(d_p + d_{p-1}\right)^2 \le N$

If $\left(d_p + d_{p-1}\right)^2 = N$ then,

$$\sqrt{N} = d_p + d_{p-1}$$

- Else ($\left(d_p + d_{p-1}\right)^2 < N$)

Find the biggest simple number d_{p-2} such that: $\left(d_p + d_{p-1} + d_{p-2}\right)^2 \le N$

At the end, we find,

$$\sum_{i=0}^{i=p} d_i = \sqrt{N}$$

7.2. The table method.

Al-Nīsābūrī draws a table whose columns number is equal to the number of digits of the integer he wants to extract its square root, and then he places those digits on the top of the table from right to left. Next, he puts a mark (dot) atop the first digit (first column) then he skips the next digit until he reaches the last mark (on the last digit or the digit before). Al-Nīsābūrī uses three rows: the upper row is the root row, the row in the middle is the number row and the lower row is the side row.

- Al-Nīsābūrī's example[1] to calculate the square root of 104976 $\left[\sqrt{104976}\right]$ (Abbassi, A. (2010), pp. 59- 60).

1. We draw a table as we mentioned before, it would be like this:

Table 1. Calculate the square root

	.		.		.
1	**0**	**4**	**9**	**7**	**6**

2. We find the greatest simple number c that would be placed on the last mark and under it in the side row such that: $c^2 \leq 10$ and hence: $c = 3$.

After that we subtract its square from 10, it would be 1, then we add 3 and 3 and we place the total 6 in the column on the right as follows:

Table 2. Calculate the square root

	3				
	.		.		.
1	**0**	**4**	**9**	**7**	**6**
	1				
	3	6			

3. We find the greatest simple number d to place on the second mark and under it in the side row such that,

$$\overline{6d} \times d \leq 149$$

And hence, $d = 2$; the difference would be placed in the number row $149 - 124 = 25$.

Then we add 2 and 2, and we place the total 64 in the column on the right and we get the following

[1] Throughout the rest, the symbols are written in the current form.

table:

Table 3. Calculate the square root

	3		2		
	.		.		.
1	**0**	**4**	**9**	**7**	**6**
	1	2	5		
	3	6	2		
			6	4	

4. We find the greatest simple number k that would be placed on the first mark and under it in the side row, such that,

$$k \times \overline{64k} \leq 2576$$

And hence, $k = 4$ we place the found 4 on the first mark and under it as we have said before. The difference at the end of this process is 0; and therefore, this is a square number, and its square root is 324.

Table 4. Calculate the square root

	3		2		4
	.		.		.
1	**0**	**4**	**9**	**7**	**6**
	1	2	5		
					0
	3	6	2		
			6	4	4

This number is also, *munṭaq* (perfect square); if the difference is not zero then this number is *aṣamm* (imperfect square). Al-Nīsābūrī keeps the details of the calculation inside the table so that it could be possible to follow the steps well. His work is based on the aforementioned identity.

Both al-Nasawī (H. Suter. (1986), pp. 114-115) and al-Baghdādī (Al-Baghdādī. (1985), pp. 73-75) used this algorithm in which they double the number a, while al-Nīsābūrī and al-Ṭūsī adds a to a; hence its importance is reflected during the search of the cube root or n^{th} root because the addition operation it is that which corresponds to the method recently developed (after the 11th century) by Arabs mathematicians.

Al-Ṭūsī's work is similar, but with some differences in tools and names; unlike al-Nīsābūrī, who uses tables that show all the work steps, al-Ṭūsī does not display his calculation details, due to the use of erasure. Because al-Ṭūsī erases and replaces digits as he works, his final table does not show the calculations that took place during the work.

Perhaps the Indian method of calculation affected him because it bears the title of his Treatise. As we know the Indian methode is called the "al-Takht wa al-Turāb"(Board and Dust) method and it is based on the use of erasure and displacement, a feature of most of the Arabic works on calculation composed before the 13th century. In al-Nīsābūrī' work, we find a paradigm shift in the abandonment of erasure.

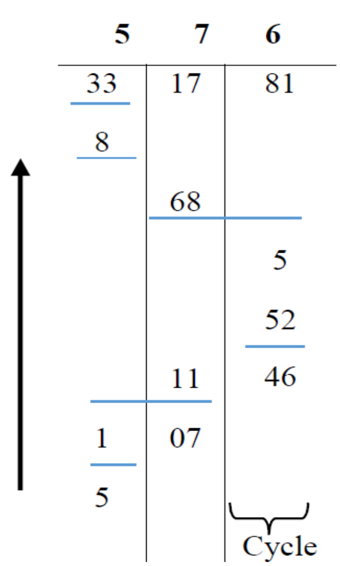
Significant similarity exists between al-Nīsābūrī's work and al-Kāshī's in his Treatise *Miftāḥ al-ḥisāb* (Key to Arithmetic). Al-Ṭūsī names a perfect square *munṭaq* as (open number) and other number *aṣamm* (deaf number).

The method of the square root of al-Nīsābūrī is the same as al-Ṭūsī's, but we notice that al-Kāshī and al-Nīsābūrī use the method by explaining the stages of work, without erasing the intermediate stages.

The al-Kāshī method begins by dividing the digits of the number proposed to extract its square root into pairs called "cycles", starting from the right.

For al-Kāshī, the row of the root (work begins from bottom to top) (Berggren, J. L. (2016), pp. 57-72); but al-Nīsābūrī does the opposite (top to bottom)

Al-Kāshī's example to calculate $\sqrt{331781}$. (Abbassi A. (2010), pp. 78-78).



7.3. Square root approximation in *al- Shamsiyya*.

Supposing that, $N = n^2 + r < (1+n)^2$, n an integer and $r \neq 0$, then the following approximation can be used,

$$\sqrt[2]{N} = \sqrt[2]{n^2 + r} \cong n + \frac{r}{2n+1} \qquad (5.2)$$

Al-Nasawī and al- Baghdādī use this formula also used by al-Uqlīdisī (Al-Uqlīdisī. (1984), p. 498), Kūshyār ibn Labbān (d. ca. 1000) (Djebbar A. (1987)) and al-Karajī (d. 1029) (Al-Karajī. (1986), p. 168). This approximation could also be found in al-Ṭūsī's work who called this amount $(2n+1)$ (M. Régis & Roshdi R. (1997), p. 383) "conventional denominator" and later in al-Kāshī's work.

Ibn al-Bannāʾ from the Islamic Maghreb in his book *Talkhīṣ ʿamal al-ḥisāb* (Summary of arithmetical operations) formulates the following approximation (Lamrabet, D. (1994), p. 195),

$$\sqrt[2]{n^2 + r} \cong \begin{cases} n + \frac{r}{2n}, & r \leq n \\ n + \frac{r+1}{2(n+1)}, & r > n \end{cases} \qquad (5.3)$$

Ibn al-Bannāʾ stated this formula in the *Talkhīṣ* using the phrase "refinement of approximation".

8. Extracting roots of higher-order

Using the same ideas and tools, al-Nīsābūrī generalizes the square root extraction method to any other higher-order root by writing the number into a decimal system then using an expansion of the famous binomial,

$$(a+b)^n.$$

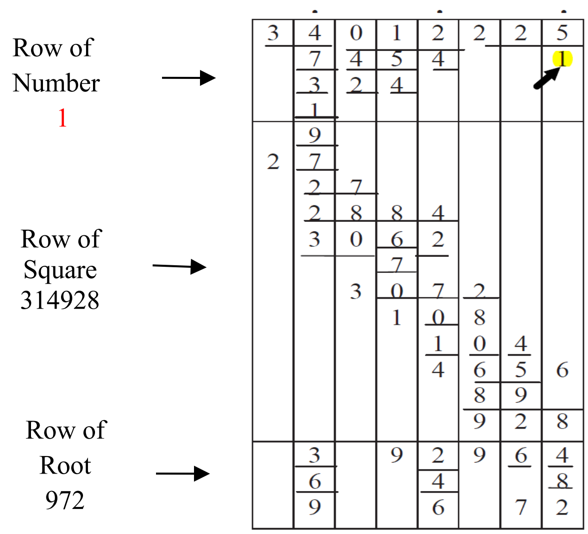
In case the difference is not zero, one is added to the total of all results in lines except the number row to get the denominator of the different fraction. Hence the expression of approximation,

$$\sqrt[m]{n^m + r} \cong n + \frac{r}{(n+1)^m - n^m} \qquad (5.4)$$

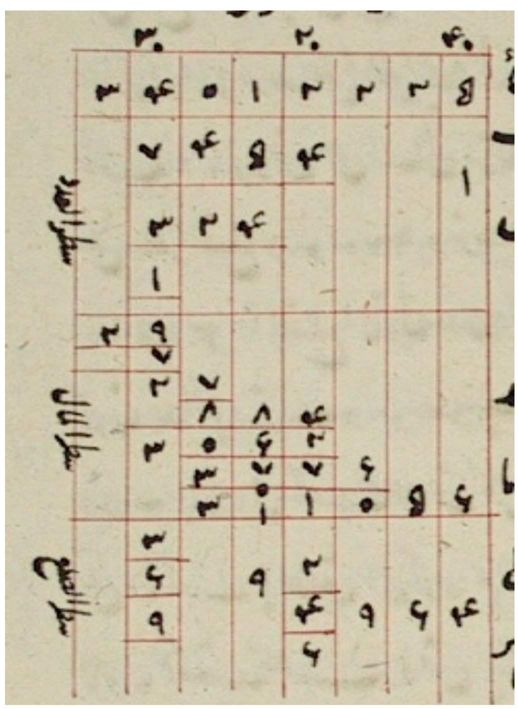
With m and k two integers.

Table 5 presents, al-Nīsābūrī's example to calculate $\sqrt[3]{34012225}$ (Abbassi, A. (2010), p. 70).

Table. 5. Calculate the cube root



$$\sqrt[3]{34012225} \cong 324 + \frac{1}{314928 + 972 + 1} = 324 + \frac{1}{315901}$$



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Al-Ṭūsī, to find the cube root, uses the following order of operations, as shown in the four rows below (Al-Uqlīdisī. (1984), pp. 507-508)

Row of Result	→	a
Row of Number	→	The number
Row of Square	→	$a^2 \quad 3a^2$
Row of Root	→	$a \quad 2\,a \quad 3a$

This is what he calls "*uṣūl al- manāẓil*" that is the coefficients in the binomial expression$(a+b)^n$. The difference between Ṭūsī's work and al-Nīsābūrī's work lies in some working processes; for instance, al-Nīsābūrī uses a different way in writing coefficients of $(a+b)^n$; he prefers the table method; he shows all calculation details unlike al-Ṭūsī who uses erasure; al-Ṭūsī defines the "conventional denominator" for example D as,

$$D = (n+1)^m - n^m \qquad (5.5)$$

This equals the sum of all results in all rows except the number row plus one. Then al-Ṭūsī introduces the approximation formula (Djebbar, A. (1987)):

$$\sqrt[m]{n^m + r} \approx n + \frac{r}{\sum_{k=1}^{m-1} c_m^k n^k + 1}$$

$$= n + \frac{r}{(n+1)^m - n^m} \qquad (5.6)$$

Al-Ṭūsī provides the following example (Youschkevitch A. P. (1976), p. 80) $\sqrt[6]{244140626}$.

$$\sqrt[6]{244140626} \cong 25 + \frac{1}{26^6 - 25^6} = 25 + \frac{1}{64775151}$$

Al-Ṭūsī uses a table in which he presents binomial coefficients for $n = 12$, to be used in extracting roots of higher-order (Al-Tūsī Ms, n° 973, fol. 51).

Proving the previous approximation formula is possible by using linear polarisation methods (Djebbar, A. (1980), p. 35) which had been known among the eleventh-century astronomers; based on observation, by setting,

$$\sqrt[m]{n^m + r} \cong n + \mu \qquad \text{such that, } 0 \le \mu < 1$$

Then we put, $\mu^m \sim \mu^{m-1} \sim \ldots \sim \mu^3 \sim \mu^2 \sim \mu$ so,

$$\mu = \frac{r}{\sum_{k=1}^{m-1} c_m^k n^k + 1}$$

9. Al-Nisaburi's rules for calculating roots (Abbassi A. (2010), pp. 71-72)**.**

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{a \times b}$$

$$\sqrt[n]{a} = \frac{\sqrt[n]{a.b^n}}{b}$$

$$\sqrt[n]{\frac{a}{b}} = \sqrt[n]{\frac{a.b^{n-1}}{b^n}} = \frac{\sqrt[n]{a.b^{n-1}}}{b}$$

$$\sqrt[n]{\frac{a^n}{b^n}} = \frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}} = \frac{a}{b}$$

Al-Nīsābūrī uses these rules to simplify expressions or approximate a root. For examples,

1. $\sqrt[2]{2} = \sqrt{1+1} \cong 1 + \frac{1}{3}$ according to (5.2)

On the other hand,

$$\sqrt[2]{2} = \frac{\sqrt{200}}{10} = \frac{\sqrt{196+4}}{10} \cong \frac{14 + \frac{4}{29}}{10} = 1 + \frac{12}{29}$$

More precise root for $\sqrt{2}$. (Abbassi, (2010), p. 62).

2.

$$\sqrt[4]{2 + \frac{1}{2}} = \sqrt[4]{\frac{5}{2}} = \frac{\sqrt[4]{5 \times 2^3}}{\sqrt[4]{2^4}} = \frac{\sqrt[4]{40}}{2}$$

But,

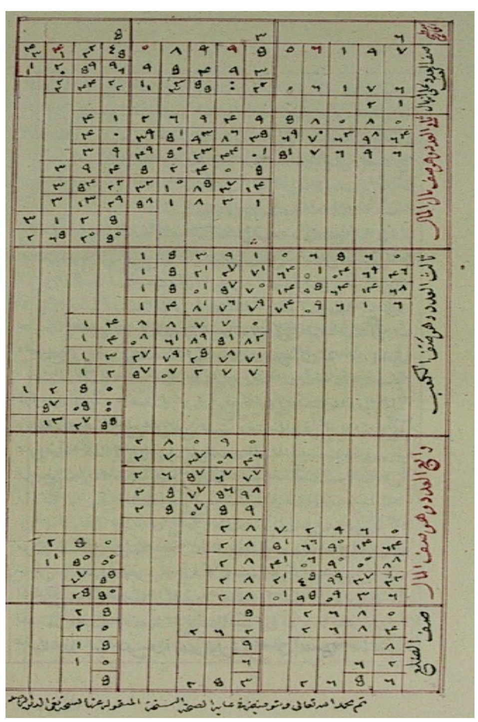
$$\frac{\sqrt[4]{40}}{2} \cong \frac{2 + \frac{24}{3^4 - 2^4}}{2} = 2 + \frac{24}{65}$$

(Abbassi, (2010), p. 72).

Al-Uqlīdisī, Kūshyār ibn Labbān (10th century), al-Nasawī and al-Baghdādī are indicating a process for the extraction of cube roots. (B. G. Johansson. (2011), p. 340).

Al-Kāshī provides an example of extracting the 5th root of the number 44240899506197 by using tables and basing on what is currently known as Horner's plan; he uses the differences of $(a+b)^n - b^n$ to get the following result by using the previous approximation,

$$\sqrt[5]{44240899506197} \cong 536 + \frac{21}{414237740281}$$



Al-Kāshī, Ms. *Miftāḥ al-ḥisāb.* Istanbul, ESAD EFENDI, n. 03175, fol. 13a.

10. Conclusions.

As we noted the similarities between the works of al-Kāshī and al-Nīsābūrī, the differences appear in some technical aspects such as the table's layout and setting numbers where al-Kāshī works from bottom to top when you fill in the table.

-The method of extraction of the n^{th}root of al-Kāshī and al-Nīsābūrī uses a method when none of the intermediate steps are erased.

-There is no proof in the work of al-Nīsābūrī; also, al-Nīsābūrī gives only one example of extraction of the cube root in the case of the roots of higher-order.
- We can therefore consider the production of al-Nīsābūrī as an intermediate work between the works of al-Ṭūsī and al-Kāshī for the extraction of the root n^{th}generally.

The importance of al-Nīsābūrī's work lies in the fact that it represents a link between what was produced in this field by al-Ṭūsī in his book<<*Jawāmi al-Hisāb bi al-Takht wa al-Turāb*>> and by al-Kāshī in his Treatise <<*Miftāḥ al-ḥisāb*>> in terms of tools and techniques and the use of erasure or dispensing with it.

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