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Three new weaker notions of fuzzy open sets and related covering concepts

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Abstract

A subset A of an ordinary topological space (X, T) is ω-open (resp. $\mathcal{N}$-open) if for each $x \in A$, there exists $U \in T$ such that $x \in U$ and $U - A$ is countable (resp. finite). In this work, we extend ω-open and $\mathcal{N}$-open notions to include L-topological spaces, where L is an F-lattice, and we introduce a third notion of L-sets weaker than both of them. For a given L-topological space, the new notions give us three new finer L-topological spaces, which can help us to increase our understanding of this L-topological space. By means of these new notions in L-topological spaces, several types of Chang's compactness, and Wong's Lindelöfness will be introduced. We make many comparisons between the new notions, and between them and some other related concepts. Several characterizations of the new concepts are given and two characterizations of Wong's Lindelöfness concept are given.

Keywords: Fuzzy compactness; fuzzy Lindelöfness; L-topology; Q-neighborhood; ω-open sets.

1. Introduction

Dealing with the fuzzy set theory is still a hot area of research in almost all branches of mathematics and computer science (Chauhan *et al*., 2014; Davvaz & Leoreanu-Fotea, 2014; Davvaz & Hassani Sadrabadi, 2014; Et *et al*., 2014; Pant *et al*., 2015; Sen & Roy, 2013; Zulfiqar, 2014; Zulfiqar & Shabir, 2015). The fuzzy topology as an important part of the fuzzy set theory has been significantly developed in the last years. The best outlook of this development can be seen in (Höhle, 1991; Höhle *et al*., 1995; Höhle & Šostak, 1999; Höhle, 2001; Liu & Luo, 1997; Rodabaugh *et al*., 1992; Rodabaugh & Klement, 2003; Wang, 1988), in which several definitions of fuzzy topology appear. Throughout this paper, our definition of fuzzy topology will be the one which appears in (Höhle & Šostak, 1999) with the notation 'L-topology', where L is an F-lattice.

Let (X, T) be an ordinary topological space and let A be an ordinary subset of X. A point $x \in X$ is called a condensation point of A if for each $U \in T$ with $x \in U$, the set $U \cap A$ is uncountable. A is called ω-closed (Hdeib, 1982) if it contains all its condensation points. The complement of an ω-closed set is called ω-open. It is known that A is ω-open if and only if for each $x \in A$, there exists $U \in T$ and a countable subset $C \subset X$ such that $x \in U - C \subset A$. A is called $\mathcal{N}$-open (Al-Omari & Noorani, 2009) if for each $x \in A$, there exists $U \in T$ such that $x \in U$ and $U - A$ is finite.

Using ω-open sets, Lindelöfness has been characterized in (Hdeib, 1982), several continuity concepts have been introduced and studied in (Al-Hawary & Al-Omari, 2006a; Al-Omari & Noorani, 2007b; Al-Omari *et al*., 2009b; Hdeib, 1989), and several generalizations of paracompactness have been introduced and studied in (Al Ghour, 2006). Also, some modifications of both ω-open and ω-closed sets appear in (Al-Hawary & Al-Omari, 2006b; Al-Omari & Noorani, 2007a; Al-Zoubi, 2005; Sarsak, 2003). The authors in (Al-Omari *et al*., 2009a; Al-Omari & Noorani, 2009) have characterized compactness and strong compactness using $\mathcal{N}$-open sets. The door is still open to use ω-open ($\mathcal{N}$-open) sets for the purpose of generalizing some known concepts or improving some known results.

When we define a reasonable generalization of open L-sets in L-topological spaces, we hope that this will open the door for a number of future related studies. For example, as a generalization of open sets in I-topological spaces, semiopen sets were defined in (Azad, 1981) for the reason of introducing some continuity concepts, then many related research articles appeared, for instance, (Cho & Lee, 2005; Dang *et al*., 1994; Ganguly & Saha, 1986; Ghosh, 1990; Mukherjee & Sinha, 1989a; 1989b; Mukherjee, 1999). In this work, for the purposes of introducing and studying new types of compactness and

Lindelöfness in L-topological spaces, we will extend the notions ω-open and $\mathcal{N}$-open sets to include L-topological spaces, and we will introduce a notion which is weaker than ω-openness.

2. Preliminaries

In this section, we will introduce concepts, symbols, and results, which will be used in the sequel. For non-cited things, we refer the reader to (Höhle & Šostak, 1999) and (Liu & Luo, 1997).

Throughout this paper, L is an F-lattice; i.e., a completely distributive lattice with an order-reversing involution $'$ on it, and with smallest element 0 and largest element $1\,(0 \neq 1)$. $c \in L$ is called a molecule of L, if $c \neq 0$ and for arbitrary $a, b \in L$, $c \leq a \vee b$ implies $c \leq a$ or $c \leq b$. $M(L)$ will denote the set of all molecules of L. Let X and Y be two ordinary nonempty sets. In this paper, I will denote the complete distributive lattice [0, 1] with the usual order and order-reversing involution $'$, $r' = 1 - r$ for every $r \in I$. An L-subset of X is a function with domain X and values in L; i.e., an element of L^X. L^X under the pointwise ordering:

for $A, B \in L^X, A \leq B$ in L^X if and only if $A(x) \leq B(x)$ in for all $x \in X$ $A'(x) = (A(x))'$ for all $x \in X$ is also an F-lattice.

If $r \in L$ then r_X denotes the L-set given by $r_X(x) = r$ for all $x \in X$; i.e., r_X denotes the "constant" L-set of level r; i.e., the smallest and the largest elements of L^X are denoted respectively by 0_X and 1_X. $A \in L^X$ is called an L-crisp subset on X if there exists an ordinary subset $G \subset X$ such that $A = \chi_G : X \to \{0,1\} \subset L$, i.e., if A is a characteristic function of some ordinary subset of X. An L-point on X is an L-subset $x_a \in L^X$ defined as follows:

$$\text{for every } y \in X, x_a(y) = \begin{cases} a & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

$\mathrm{Pt}(L^X)$ will denote the set of all L-points on X. An L-point $x_a \in \mathrm{Pt}(L^X)$ is said to belong to an L-set $A \in L^X$ (notation: $x_a \in A$) if and only if $a \leq A(x)$. It is well known that $M(L^X) = \{x_a : x \in X, a \in M(L)\}$.

An L-topological space or L-ts for short, is a pair (L^X, δ), where X is a nonempty set, L is an F-lattice and δ called an L-topology on X is a subfamily of L^X satisfying the following three axioms:

(i) $0_X, 1_X \in \delta$.

(ii) If $U, V \in \delta$, then $U \wedge V \in \delta$.

(iii) If $\{U_j : j \in J\} \subset \delta$, then $\bigvee \{U_j : j \in J\} \in \delta$.

The elements of δ are called open L-sets. An L-set A is called closed if $A' \in \delta$.

Let (L^X, δ) be an L-ts, $\delta_\circ \subset \delta$. $\delta_\circ$ is called a base of δ, if $\delta = \{\bigvee \mathcal{A} : \mathcal{A} \subset \delta_\circ\} \cup \{0_X\}$. $\delta_\circ$ is called a subbase of δ, if $\{\bigwedge \mathcal{A} : \mathcal{A} \subset \delta_\circ$ and $\mathcal{A}$ is a nonempty finite set$\}$ forms a base of δ. For every $x_a \in \mathrm{Pt}(L^X)$ and $A \in L^X$, we say x_a is quasi-coincident with A, denoted by $x_a \ll A$, if $x_a \nleq A'$. $U \in \delta$ is called a quasi-coincident neighborhood of x_a in (L^X, δ), shorted as Q-neighborhood, if $x_a \ll U$. The family of all Q-neighborhoods of x_a in (L^X, δ) is denoted by $\mathcal{Q}(x_a)$.

Throughout this paper, for an L-subset A of an L-ts (L^X, δ), the support of A is defined by $\{x \in X : A(x) > 0\}$ and is denoted by $Supp(A)$. For any nonempty set X, X_{coc} (resp. X_{cof}, X_{disc}) will denote the ordinary cocountable (resp. cofinite, discrete) topology on X, and τ^{scoc} (resp. τ^{scof}, τ^{sdisc}) will denote the L-topology $\{\mathcal{X}_G : G \in X_{coc}\}$ (resp. $\{\mathcal{X}_G : G \in X_{cof}\}, \{\mathcal{X}_G : G \in X_{disc}\}$) on X. For an L-ts (L^X, δ), denote the families $\{U \wedge \mathcal{X}_G : U \in \delta, G \in X_{coc}\}, \{U \wedge \mathcal{X}_G : U \in \delta, G \in X_{cof}\}$, and $\{U \wedge \mathcal{X}_{\{x\}} : U \in \delta, x \in X\}$ by δ_{coc}, δ_{cof}, and δ_{disc}, respectively.

The following definitions and proposition will be used in the sequel:

Definition 2.1. An L-ts (L^X, δ) is called T_c, if for every $x \in X$, the L-crisp subset $\mathcal{X}_{\{x\}}$ is L-closed.

The author in (Fora, 1989) defined the concept T_c for the special F-lattice I.

Definition 2.2. An L-ts (L^X, δ) is called P-L-ts, if the meet of any countable family of L-open subsets of (L^X, δ) is L-open.

The author in (Al-Hawary, 2008) defined the concept P-L-ts for the special F-lattice I.

Definition 2.3. (Liu & Luo, 1997) Associated with a given L-ts (L^X, δ) and arbitrary nonempty ordinary subset Y of X, define the induced L-topology on Y or the relative L-topology on Y by $\delta_{|Y} = \{U_{|Y} : U \in \delta\}$.

Definition 2.4. (Liu & Luo, 1997) Let (L^X, δ) be an L-ts, $A \in L^X, \mathcal{A}, \mathcal{B} \subset L^X$. $\mathcal{A}$ is called a cover of A, if $\bigvee \mathcal{A} \geq A$; particularly, $\mathcal{A}$ is called a cover of (L^X, δ), if $\mathcal{A}$ is a cover of 1_X. $\mathcal{A}$ is called an open cover of A, if $\mathcal{A} \subset \delta$ and $\mathcal{A}$ is a cover of A. For a cover $\mathcal{A}$ of A, $\mathcal{B}$ is called a subcover of $\mathcal{A}$, if $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{B}$ is still a cover of A.

Proposition 2.5. (Liu & Luo, 1997) Let (L^X, δ) be an L-ts, $\mathcal{A} \subset \delta$. Then the following conditions are equivalent:

(i) $\mathcal{A}$ is a base of δ.

(ii) For all $x_a \in M(L^X)$, for all $U \in \mathcal{Q}(x_a)$, there exists $A \in \mathcal{A} \cap \mathcal{Q}(x_a)$ such that $A \leq U$.

3. !-Open, N-open, D-open *L*-sets

We start by defining the main notions of this paper.

Definition 3.1. Let (L^X, δ) be an L-ts and let $W, A \in L^X$.

(i) W is called an ω-open (resp. $\mathcal{N}$-open) L-set in (L^X, δ), if for every $x_a \in M(L^X)$ with $x_a \ll W$, there exist $U \in \mathcal{Q}(x_a)$ and $G \in X_{coc}$ (resp. $G \in X_{cof}$) such that $x \in G$ and $U \wedge \mathcal{X}_G \leq W$.

(ii) W is called a $\mathcal{D}$-open L-set in (L^X, δ) if for every $x_a \in M(L^X)$ with $x_a \ll W$, there exists $U \in \mathcal{Q}(x_a)$ such that $U \wedge \mathcal{X}_{\{x\}} \leq W$.

(iii) A is called an ω-closed (resp. an $\mathcal{N}$-closed, a $\mathcal{D}$-closed) L-set in (L^X, δ) if A' is an ω-open (resp. an $\mathcal{N}$-open, a $\mathcal{D}$-open) L-set in (L^X, δ).

For an L-ts (L^X, δ) denote the family of all ω-open (resp. $\mathcal{N}$-open, $\mathcal{D}$-open) L-sets in (L^X, δ) by δ_ω (resp. $\delta_\mathcal{N}$, $\delta_\mathcal{D}$).

The following theorem summarizes several important properties and relationships related to the notions in Definition 3.1.

Theorem 3.2. Let (L^X, δ) be an L-ts. Then the following conclusions hold.

(i) $\delta \subset \delta_{cof} \subset \delta_{coc} \subset \delta_\omega \subset \delta_\mathcal{D}$.

(ii) $\delta_{cof} \subset \delta_\mathcal{N}$.

(iii) $\delta_{disc} \subset \delta_\mathcal{D}$.

(iv) Each of δ_ω, $\delta_\mathcal{N}$, and $\delta_\mathcal{D}$ forms an L-topology on X.

(v) δ_{coc} (resp. $\delta_{cof}, \delta_{disc}$) forms a base of the L-topology δ_ω (resp. $\delta_\mathcal{N}$, $\delta_\mathcal{D}$) on X.

(vi) $\delta \cup \tau^{scoc}$ (resp. $\delta \cup \tau^{scof}$, $\delta \cup \tau^{sdisc}$) forms a subbase of the L-topology δ_ω (resp. $\delta_\mathcal{N}$, $\delta_\mathcal{D}$) on X.

(vii) $\delta = \delta_\omega$ (resp. $\delta = \delta_\mathcal{N}$, $\delta = \delta_\mathcal{D}$) if and only if $\tau^{scoc} \subset \delta$ (resp. $\tau^{scof} \subset \delta, \tau^{sdisc} \subset \delta$).

(viii) $\delta_\mathcal{N} \subset \delta_\omega$.

Proof. (i) For every $U \in \delta$, $U = U \wedge \mathcal{X}_X \in \delta_{cof}$, hence $\delta \subset \delta_{cof}$. That $\delta_{cof} \subset \delta_{coc}$ is obvious. To see that $\delta_{coc} \subset \delta_\omega$, let $U \in \delta$, $G \in X_{coc}$, and $x_a \in M(L^X)$ with $x_a \ll U \wedge \mathcal{X}_G$, it is not difficult to see that $x \in G$. Thus, we have $U \in \mathcal{Q}(x_a)$, $G \in X_{coc}$, $x \in G$, and $U \wedge \mathcal{X}_G \leq U \wedge \mathcal{X}_G$, which shows that $U \wedge \mathcal{X}_G \in \delta_\omega$. To see that $\delta_\omega \subset \delta_\mathcal{D}$, let $W \in \delta_\omega$ and let $x_a \in M(L^X)$ with $x_a \ll W$. Then there exist $U \in \mathcal{Q}(x_a)$ and $G \in X_{coc}$ such that $x \in G$ and $U \wedge \mathcal{X}_G \leq W$ and hence $U \wedge \mathcal{X}_{\{x\}} \leq U \wedge \mathcal{X}_G \leq W$, consequently, $W \in \delta_\mathcal{D}$.

(ii) and (iii) Each similar to that used in the proof of $\delta_{coc} \subset \delta_\omega$.

(iv) Only we prove δ_ω satisfies the conditions of L-topology, the others are similar:

(1) By (i), $0_X, 1_X \in \delta_\omega$.

(2) Let $W, Q \in \delta_\omega$ and $x_a \in M(L^X)$ with $x_a \ll W \wedge Q$. Then $x_a \ll W$ and $x_a \ll Q$, and so there exist $U, V \in \mathcal{Q}(x_a)$ and $G, H \in X_{coc}$ such that $x \in G \cap H$, $U \wedge \mathcal{X}_G \leq W$, and $V \wedge \mathcal{X}_H \leq Q$. Thus, we have $U \wedge V \in \mathcal{Q}(x_a)$, $G \cap H \in X_{coc}$, $x \in G \cap H$, and $(U \wedge V) \wedge \mathcal{X}_{G \cap H} = (U \wedge \mathcal{X}_G) \wedge (V \wedge \mathcal{X}_H) \leq W \wedge Q$, hence $W \wedge Q \in \delta_\omega$.

(3) Let $\{W_\alpha : \alpha \in \Lambda\} \subset \delta_\omega$ and $x_a \in M(L^X)$ with $x_a \ll \bigvee \{W_\alpha : \alpha \in \Lambda\}$. There exists $\beta \in \Lambda$ such that $x_a \ll W_\beta$. Thus, there exist $U \in \mathcal{Q}(x_a)$ and $G \in X_{coc}$ such that $x \in G$ and $U \wedge \mathcal{X}_G \leq W_\beta \leq \bigvee \{W_\alpha : \alpha \in \Lambda\}$, hence $\bigvee \{W_\alpha : \alpha \in \Lambda\} \in \delta_\omega$.

(v) Since by (i) (resp. (ii), (iii)) $\delta_{coc} \subset \delta_\omega$ (resp. $\delta_{cof} \subset \delta_\mathcal{N}$, $\delta_{disc} \subset \delta_\mathcal{D}$), then by Proposition 2.5, it is easy to check that δ_{coc} (resp. δ_{cof}, δ_{disc}) forms a base of δ_ω (resp. $\delta_\mathcal{N}$, $\delta_\mathcal{D}$).

(vi) Follows directly from (v).

(vii) Follows directly from (vi).

(viii) Since by (i), $\delta_{cof} \subset \delta_{coc}$, then by (v), it follows that $\delta_\mathcal{N} \subset \delta_\omega$.

For an L-ts (L^X, δ), each of the sequence of the inclusions $\delta \subset \delta_\mathcal{N} \subset \delta_\omega \subset \delta_\mathcal{D}$ cannot be replaced by equality, in general. As an example, take X as an uncountable set, L as an arbitrary F-lattice, $\delta = \{0_X, 1_X\}$. Then $\delta \neq \delta_\mathcal{N}$, $\delta_\mathcal{N} \neq \delta_\omega$, and $\delta_\omega \neq \delta_\mathcal{D}$. In the first part of the rest of this section, we study conditions which are sufficient to have equality between two of $\delta, \delta_\mathcal{N}, \delta_\omega, \delta_\mathcal{D}$ or more.

Although an L-ts (L^X, δ) need not to be T_c, in general, the following theorem says that each of the new L-ts' $(L^X, \delta_\mathcal{N}), (L^X, \delta_\omega), (L^X, \delta_\mathcal{D})$ is T_c.

Theorem 3.3. For any L-ts (L^X, δ), each of $(L^X, \delta_\mathcal{N})$, $(L^X, \delta_\omega), (L^X, \delta_\mathcal{D})$ is T_c.

Proof. As by Theorem 3.2 (i) and (viii) $\delta_{\mathcal{N}} \subset \delta_{\omega} \subset \delta_{\mathcal{D}}$, it is sufficient to see that $(L^X, \delta_{\mathcal{N}})$ is T_c. Let $x \in X$. Then $\mathcal{X}_{X-\{x\}} \in \delta_{\mathcal{N}}$, and hence $(\mathcal{X}_{X-\{x\}})' = \mathcal{X}_{\{x\}}$ is a closed L-set in $(L^X, \delta_{\mathcal{N}})$.

In a T_c L-ts, the concepts $\mathcal{N}$-open and open L-sets are equivalent, as the following result shows.

Theorem 3.4. An L-ts (L^X, δ) is T_c if and only if $\delta = \delta_{\mathcal{N}}$.

Proof. Necessity. Suppose (L^X, δ) is T_c. For every $G \in X_{cof}$,

$$(\mathcal{X}_G)' = \bigvee \{\mathcal{X}_{\{x\}} : x \in X - G\}$$

is a closed L-set in (L^X, δ), and hence $\mathcal{X}_G \in \delta$. Therefore, by Theorem 3.2 (vii), it follows that $\delta = \delta_{\mathcal{N}}$.

Sufficiency. Follows since by Theorem 3.3, $(L^X, \delta_{\mathcal{N}})$ is T_c.

For an L-ts (L^X, δ), in Theorem 3.4, we see that the property T_c is a sufficient (and necessary) condition for $\delta = \delta_{\mathcal{N}}$. In the following theorem, we add the condition that '(L^X, δ) P-L-ts' to get the stronger result, $\delta = \delta_{\omega}$.

Theorem 3.5. If (L^X, δ) is T_c and P-L-ts, then $\delta = \delta_{\mathcal{N}} = \delta_{\omega}$.

Proof. Since (L^X, δ) is P-L-ts, then for every $G \in X_{cof}$,

$$(\mathcal{X}_G)' = \bigvee \{\mathcal{X}_{\{x\}} : x \in X - G\}$$

is a closed L-set in (L^X, δ), and hence $\mathcal{X}_G \in \delta$. Therefore, by Theorem 3.2 (vii), it follows that $\delta = \delta_{\omega}$. This ends the proof because by Theorem 3.2 (i) and (viii), we have $\delta \subset \delta_{\mathcal{N}} \subset \delta_{\omega}$.

Definition 3.6. An L-ts (L^X, δ) is said to be CS (FS) if δ has a base $\mathcal{A}$ such that for every $A \in \mathcal{A}$, *Supp* (A) is countable (finite).

The property '(L^X, δ) CS' (resp. '(L^X, δ) is FS') is sufficient for $\delta_{\mathcal{D}} = \delta_{\omega}$ (resp. $\delta_{\mathcal{N}} = \delta_{\omega} = \delta_{\mathcal{D}}$), as the following theorem says.

Theorem 3.7. Let (L^X, δ) be an L-ts. Then

(i) If (L^X, δ) is CS, then $\delta_{\mathcal{D}} = \delta_{\omega}$.

(ii) (L^X, δ) is FS, then $\delta_{\mathcal{N}} = \delta_{\omega} = \delta_{\mathcal{D}}$.

Proof. (i) By Theorem 3.2 (i), $\delta_{\omega} \subset \delta_{\mathcal{D}}$. We show that $\delta_{\mathcal{D}} \subset \delta_{\omega}$. Let $W \in \delta_{\mathcal{D}}$ and let $x_a \in M(L^X)$ with $x_a \ll W$. Then there exists $U \in \mathcal{Q}(x_a)$ such that $U \wedge \mathcal{X}_{\{x\}} \leq W$. As (L^X, δ) is CS, there exists $V \in \mathcal{Q}(x_a)$ such that $V \leq U$ and $Supp\,(V)$ is countable. Put $G = (X - Supp\,(V)) \cup \{x\}$. Therefore, $G \in X_{coc}$, $x \in G$, and $V \wedge \mathcal{X}_G = V \wedge \mathcal{X}_{\{x\}} \leq U \wedge \mathcal{X}_{\{x\}} \leq W$. Hence, $W \in \delta_{\omega}$.

(ii) By imitating the proof of (i), one can show that $\delta_{\mathcal{N}} = \delta_{\mathcal{D}}$.

Corollary 3.8. Let (L^X, δ) be an L-ts. Then

(i) If X is countable, then $\delta_{\mathcal{D}} = \delta_{\omega}$.

(ii) If X is finite, then $\delta_{\mathcal{N}} = \delta_{\omega} = \delta_{\mathcal{D}}$.

The following lemma will be used in the proof of the next two theorems.

Lemma 3.9. Let (L^X, δ) be an L-ts, A a base of δ. Then

(i) $\{A \wedge \mathcal{X}_G : A \in \mathcal{A}$ and $G \in X_{coc}\}$ is a base of δ_{ω}.

(ii) $\{A \wedge \mathcal{X}_G : A \in \mathcal{A}$ and $G \in X_{cof}\}$ is a base of $\delta_{\mathcal{N}}$.

Proof. (i) Since by Theorem 3.2 (v), δ_{coc} forms a base of δ_{ω}, it is sufficient to show that every element of δ_{coc} is a join of a family of elements of $\{A \wedge \mathcal{X}_G : A \in \mathcal{A}$ and $G \in X_{coc}\}$. Let $U \wedge \mathcal{X}_G \in \delta_{coc}$ where $U \in \delta$ and $G \in X_{coc}$. Since $\mathcal{A}$ is a base of δ, there exists $\mathcal{A}^* \subset \mathcal{A}$ such that $U = \bigvee \{A : A \in \mathcal{A}^*\}$. Since L^X is distributive, $U \wedge \mathcal{X}_G = \bigvee \{A \wedge \mathcal{X}_G : A \in \mathcal{A}^*\}$. This completes the proof.

(ii) Similar to the proof of (i).

Theorem 3.10. Let (L^X, δ) be an L-ts. Then

(i) $(\delta_{\omega})_{\omega} = \delta_{\omega}$.

(ii) $(\delta_{\mathcal{N}})_{\mathcal{N}} = \delta_{\mathcal{N}}$.

Proof. (i) By Theorem 3.2 (v), δ_{coc} is a base of δ_{ω}. Thus by Lemma 3.9, it follows that $\{A \wedge \mathcal{X}_G : A \in \delta_{coc}, G \in X_{coc}\}$ is a base of $(\delta_{\omega})_{\omega}$. It is clear that $\{A \wedge \mathcal{X}_G : A \in \delta_{coc}, G \in X_{coc}\} = \delta_{coc}$. Thus, $(\delta_{\omega})_{\omega} = \delta_{\omega}$.

(ii) Similar to the proof of (i).

Theorem 3.11. Let (L^X, δ) be an L-ts. Then

(i) If (L^X, δ) is CS, then (L^X, δ_{ω}) is CS.

(ii) If (L^X, δ) is FS, then (L^X, δ_{ω}) is FS.

Proof. (i) Suppose (L^X, δ) is CS. Then there exists a base $\mathcal{A}$ of δ such that for every $A \in \mathcal{A}$, $Supp\,(A)$ is countable. So by Lemma 3.9 (i),

$$\{A \wedge \mathcal{X}_G : A \in \mathcal{A} \text{ and } G \in X_{coc}\}$$

is a base of δ_{ω}. Also, for every $A \in \mathcal{A}$ and $G \in X_{coc}$ $Supp\,(A \wedge \mathcal{X}_G) = Supp\,(A) \cap (X - G)$ is countable. This ends the proof.

ii) Similar to that used in (i).

The following theorem is natural and it will be used in the proof of the next main result:

Theorem 3.12. Let (L^X,δ) be an L-ts, $Y\subset X$, $Y\neq\emptyset$. Then

(i) $\left(\delta_{|Y}\right)_{coc}=(\delta_{coc})_{|Y}$.

(ii) $\left(\delta_{|Y}\right)_{cof}=(\delta_{cof})_{|Y}$.

(iii) $\left(\delta_{|Y}\right)_{disc}=(\delta_{disc})_{|Y}$.

Proof. The proofs of all are similar, so we prove only (i).

(i) Let $(U_{|Y})\wedge\mathcal{X}_{Y-C}\in\left(\delta_{|Y}\right)_{coc}$ where $U\in\delta$, $C\subset Y$, C is countable. Since $(U_{|Y})\wedge\mathcal{X}_{Y-C}=(U\wedge\mathcal{X}_{X-C})_{|Y}$ and $(U\wedge\mathcal{X}_{X-C})\in\delta_{coc}$, then $(U\wedge\mathcal{X}_{X-C})_{|Y}\in(\delta_{coc})_{|Y}$. Conversely, let $(U\wedge\mathcal{X}_{X-C})_{|Y}\in(\delta_{coc})_{|Y}$ where $U\in\delta$, $C\subset X$, C is countable. Since $(U\wedge\mathcal{X}_{X-C})_{|Y}=(U_{|Y})\wedge\mathcal{X}_{Y-(C\cap Y)}$, $U_{|Y}\in\delta_{|Y}$, and $C\cap Y$ is countable, then $(U\wedge\mathcal{X}_{X-C})_{|Y}\in\left(\delta_{|Y}\right)_{coc}$.

For a given L-ts (L^X,δ) and a nonempty ordinary subset $Y\subset X$, there is a strong relationship between the ω-open (resp. $\mathcal{N}$-open, $\mathcal{D}$-open) L-sets in the relative L-ts $(L^Y,\delta_{|Y})$ and ω-open (resp. $\mathcal{N}$-open, $\mathcal{D}$-open) L-sets of (L^X,δ), as the following corollary says:

Corollary 3.13. Let (L^X,δ) be an L-ts, $Y\subset X$, $Y\neq\emptyset$. Then

(i) $\left(\delta_{|Y}\right)_{\omega}=(\delta_{\omega})_{|Y}$.

(ii) $\left(\delta_{|Y}\right)_{\mathcal{N}}=(\delta_{\mathcal{N}})_{|Y}$.

(iii) $\left(\delta_{|Y}\right)_{\mathcal{D}}=(\delta_{\mathcal{D}})_{|Y}$.

In the following result, I (L) and $\mathbb{R}$ (L) will denote the L-fuzzy unit interval and the L-fuzzy real line, respectively; for some undefined symbols here we refer to (Höhle & Šostak, 1999) and (Liu & Luo, 1997).

Theorem 3.14. For every F-lattice L which has a subset $\{a,b,c\}$ satisfying $0<a<b<c<1$, $a'=c$ and $b'=b$ neither I (L) nor $\mathbb{R}$ (L) is T_c. In particular, Neither I (I) nor $\mathbb{R}$ (I) is T_c.

Proof. Consider the mappings $x:\mathbb{R}\longrightarrow L$ and $y:\mathbb{R}\longrightarrow L$ defined by

$$x(t)=\begin{cases}1 & \text{if } t\in(-\infty,0),\\ a & \text{if } t\in[0,1],\\ 0 & \text{if } t\in(1,\infty),\end{cases}$$

$$y(t)=\begin{cases}1 & \text{if } t\in(-\infty,0),\\ b & \text{if } t\in[0,1],\\ 0 & \text{if } t\in(1,\infty),\end{cases}$$

Then $[x]$, $[y]\in I$ $[L]$ with $[x]\neq[y]$. To show that I (L) is not T_c, it is sufficient to see that $\mathcal{X}_{I[L]-\{[x]\}}$ is not open in I (L). It is known that $\mathcal{A}=\{1_{I[L]}\}\cup\{L_t,R_s,L_t\wedge R_s:0\leq t,s\leq1\}$ is a base for I (L). For every t, $s\in[0,1]$, L_t $([y])=R_s$ $([y])=(L_t\wedge R_s)$ $([y])=b\neq1$. Suppose there exists $\mathcal{B}\subset\mathcal{A}$ such that $\mathcal{X}_{I[L]-\{[x]\}}=\bigvee\mathcal{B}$, then $\mathcal{X}_{I[L]-\{[x]\}}$ $([y])=1$, also, as $\mathcal{X}_{I[L]-\{[x]\}}=\bigvee\mathcal{B}$, then $\mathcal{B}\subseteq\{L_t,R_s,L_t\wedge R_s:0\leq t,s\leq1\}$ and thus $(\bigvee\mathcal{B})([y])=b\neq1$. We proved that $\mathcal{X}_{I[L]-\{[x]\}}$ is not open in I (L). Similarly, we can see that $\mathbb{R}$ (L) is not T_c.

The following result follows directly by Theorems 3.3 and 3.14.

Corollary 3.15. Let L be an F-lattice which has a subset $\{a,b,c\}$ satisfying $0<a<b<c<1,a'=c$ and $b'=b$. If I $(L)=(L^X,\delta)$ or $\mathbb{R}$ $(L)=(L^X,\delta)$, then $\delta\neq\delta_{\mathcal{N}}$, $\delta\neq\delta_{\omega}$, and $\delta\neq\delta_{\mathcal{D}}$.

The following question is natural:

If we take $x_a\in\mathrm{Pt}(L^X)$ instead of $x_a\in M(L^X)$ in Definition 3.1, what would happen?

In the following definition we take $x_a\in\mathrm{Pt}(L^X)$ instead of $x_a\in M(L^X)$ in Definition 3.1.

Definition 3.16. Let (L^X,δ) be an L-ts and let $W\in L^X$.

(i) W is called an ω^*-open (resp. $\mathcal{N}^*$-open) L-set in (L^X,δ), if for every $x_a\in\mathrm{Pt}(L^X)$ with $x_a\ll W$, there exist $U\in\mathcal{Q}(x_a)$ and $G\in X_{coc}$ (resp. $G\in X_{cof}$) such that $x\in G$ and $U\wedge\mathcal{X}_G\leq W$.

(ii) W is called a $\mathcal{D}^*$-open L-set in (L^X,δ) if for every $x_a\in\mathrm{Pt}(L^X)$ with $x_a\ll W$, there exists $U\in\mathcal{Q}(x_a)$ such that $U\wedge\mathcal{X}_{\{x\}}\leq W$.

It is clear that every ω^*-open (resp. $\mathcal{N}^*$-open, $\mathcal{D}^*$-open) L-set in an L-ts (L^X,δ) is ω-open (resp. $\mathcal{N}$-open, $\mathcal{D}$-open). In the rest of this section, we are going to show that notions ω-open (resp. $\mathcal{N}$-open, $\mathcal{D}$-open) and ω^*-open (resp. $\mathcal{N}^*$-open, $\mathcal{D}^*$-open) in an L-ts (L^X,δ) are equivalent.

For an L-ts (L^X,δ) denote the family of all ω^*-open (resp. $\mathcal{N}^*$-open, $\mathcal{D}^*$-open) L-sets in (L^X,δ) by δ_{ω^*} (resp. $\delta_{\mathcal{N}^*}$, $\delta_{\mathcal{D}^*}$).

Lemma 3.17. Let (L^X,δ) be an L-ts. Then

(i) $\delta_{coc}\subset\delta_{\omega^*}$, $\delta_{cof}\subset\delta_{\mathcal{N}^*}$ and $\delta_{disc}\subset\delta_{\mathcal{D}^*}$.

(ii) If $\{U_j:j\in J\}\subset\delta_{\omega^*}$ (resp. $\delta_{\mathcal{N}^*}$, $\delta_{\mathcal{D}^*}$), then $\bigvee\{U_j:j\in J\}\in\delta_{\omega^*}$ (resp. $\delta_{\mathcal{N}^*}$, $\delta_{\mathcal{D}^*}$).

Proof. (i) We only prove that $\delta_{coc}\subset\delta_{\omega^*}$, the others are similar:

Let $U \in \delta, G \in X_{coc}$ and let $x_a \in \text{Pt}(L^X)$ with $x_a \ll U \wedge \mathcal{X}_G$. Then $x_a \ll U$ and $x_a \ll \mathcal{X}_G$. Thus, we have $U \in \mathcal{Q}(x_a)$ and $G \in X_{coc}$ such that $x \in G$ and $U \wedge \mathcal{X}_G \leq U \wedge \mathcal{X}_G$. Hence, $U \wedge \mathcal{X}_G \in \delta_{\omega^*}$.

(ii) The proof of each is similar to that used in proving that δ_ω is closed under arbitrary join in Theorem 3.2 (iv).

Theorem 3.18. Let (L^X, δ) be an L-ts. Then $\delta_{\omega^*} = \delta_\omega$, $\delta_{\mathcal{N}^*} = \delta_{\mathcal{N}}$ and $\delta_{\mathcal{D}^*} = \delta_{\mathcal{D}}$.

Proof. We only prove that $\delta_{\omega^*} = \delta_\omega$, the others are similar:

$\delta_{\omega^*} \subset \delta_\omega$ is obvious. On the other hand, by Theorem 3.2 (v) and Lemma 3.17, we can easily seen that $\delta_\omega \subset \delta_{\omega^*}$.

4. Covering properties in L-topological spaces

We start by the following essential definition:

Definition 4.1. Let (L^X, δ) be an L-ts, $A \in L^X$, $\mathcal{A} \subset L^X$. $\mathcal{A}$ is called an ω-open (resp. $\mathcal{N}$-open, $\mathcal{D}$-open) cover of A, if $\mathcal{A} \subset \delta_\omega$ (resp. $\mathcal{A} \subset \delta_{\mathcal{N}}$, $\mathcal{A} \subset \delta_{\mathcal{D}}$) and $\mathcal{A}$ is a cover of A.

An I-ts (I^X, δ) is called compact (Chang, 1968), if every open cover of (I^X, δ) has a finite subcover.

The following definition generalizes Chang's compactness to include general L-ts'. It also uses the new kinds of L-sets to define three kinds of compactness in L-ts'.

Definition 4.2. An L-ts (L^X, δ) is called compact (resp. ω-compact, $\mathcal{N}$-compact, $\mathcal{D}$-compact) if every open (resp. ω-open, $\mathcal{N}$-open, $\mathcal{D}$-open) cover of (L^X, δ) has a finite subcover.

The following characterizations of ω-compactness, $\mathcal{N}$-compactness, and $\mathcal{D}$-compactness follow directly from the definitions:

Theorem 4.3. Let (L^X, δ) be an L-ts. Then

(i) (L^X, δ) is ω-compact if and only if (L^X, δ_ω) is compact.

(ii) (L^X, δ) is $\mathcal{N}$-compact if and only if $(L^X, \delta_{\mathcal{N}})$ is compact.

(iii) (L^X, δ) is $\mathcal{D}$-compact if and only if $(L^X, \delta_{\mathcal{D}})$ is compact.

An L-ts (L^X, δ) is neither $\mathcal{D}$-compact nor ω-compact when X is an infinite set:

Theorem 4.4. Let (L^X, δ) be an L-ts. Then

(i) If (L^X, δ) is $\mathcal{D}$-compact, then X is finite.

(ii) If (L^X, δ) is ω-compact, then X is finite.

Proof. (i) Suppose (L^X, δ) is $\mathcal{D}$-compact. Since $\{\mathcal{X}_{\{x\}} : x \in X\}$ is $\mathcal{D}$-open cover of (L^X, δ), then there exists $\{x_1, x_2, ..., x_n\} \subseteq X$ such that $\{\mathcal{X}_{\{x_i\}} : i = 1, 2, ..., n\}$ is a cover of (L^X, δ). Thus, $X = \{x_1, x_2, ..., x_n\}$ and hence X is finite.

(ii) Suppose (L^X, δ) is ω-compact and suppose to the contrary that X is infinite. If X is countable, then by Corollary 3.8 (i), $\delta_{\mathcal{D}} = \delta_\omega$ and by Theorem 4.3 (i) and (iii), it follows that (L^X, δ) is $\mathcal{D}$-compact which contradicts (i). If X is uncountable, then choose a countable infinite set $D \subseteq X$, say $D = \{x_1, x_2, ...\}$. Consider the sequence of subsets of X defined by $D_n = X - \{x_n, x_{n+1}, ...\}$, $n \in \mathbb{N}$. Then $\{\mathcal{X}_{D_n} : n \in \mathbb{N}\}$ is an ω-open cover of (L^X, δ) which has not a finite subcover. This contradicts the assumption that (L^X, δ) is ω-compact.

$\mathcal{D}$-compactness and ω-compactness are coincident to each other:

Corollary 4.5. Let (L^X, δ) be an L-ts. Then the following are equivalent:

(i) (L^X, δ) is $\mathcal{D}$-compact.

(ii) (L^X, δ) is ω-compact.

Proof. (i) $\Rightarrow$ (ii) Follows because by Theorem 3.2 (i), $\delta_\omega \subset \delta_{\mathcal{D}}$.

(ii) $\Rightarrow$ (i) Suppose (L^X, δ) is ω-compact. Then by Theorem 4.4 (ii), X is finite. Thus by Corollary 3.8, it follows that $\delta_\omega = \delta_{\mathcal{D}}$, and hence (L^X, δ) is $\mathcal{D}$-compact.

We have the following relations among these kinds of compactness in L-ts':

Theorem 4.6. The following implications hold in L-ts':

ω-compactness $\Longrightarrow$ $\mathcal{N}$-compactness $\Longrightarrow$ compactness.

Proof. Follow since by Theorem 3.2 (i), we have $\delta \subset \delta_{\mathcal{N}} \subset \delta_\omega$.

Moreover, each of the implications listed above is strict just as the following examples show:

Example 4.7. There exists an $\mathcal{N}$-compact L-ts which is not ω-compact.

Take L be any F-lattice and X be any ordinary infinite set. Then the L-ts (L^X, τ^{scof}) is clearly $\mathcal{N}$-compact. On the other hand, by Theorem 4.4 (ii), it is not ω-compact.

Example 4.8. There exists a compact L-ts which is not $\mathcal{N}$-compact.

Take $X=\{a,b\}$, $L=I$, and $\delta=\{0_X,1_X\}\cup\{t\mathcal{X}_{\{a\}}:0<t\leq 1\}$. If $\mathcal{A}$ is an open cover of (L^X,δ), then $1_X\in\mathcal{A}$, and so $\{1_X\}$ is a finite subcover of $\mathcal{A}$. Hence (L^X,δ) is compact. On the other hand, the $\mathcal{N}$-open cover $\{t\mathcal{X}_{\{a\}}:0<t<1\}\cup\{\mathcal{X}_{\{b\}}\}$ of (L^X,δ) has not any finite subcover, and hence (L^X,δ) is not $\mathcal{N}$-compact.

Remark 4.9. Example 4.8 also shows that none of the implications in Theorem 4.4 is reversible.

In an L-ts (L^X,δ) for which X is finite, ω-compactness and $\mathcal{N}$-compactness are coincident to each other:

Theorem 4.10. Let (L^X,δ) be an L-ts such that X is finite. Then the following are equivalent.

(i) (L^X,δ) is $\mathcal{D}$-compact.

(ii) (L^X,δ) is ω-compact.

(iii) (L^X,δ) is $\mathcal{N}$-compact.

Proof. Corollary 3.8 (ii) and Theorem 4.3.

The following definition will be used to give a sufficient condition for which $\mathcal{N}$-compactness and compactness are coincident to each other:

Definition 4.11. Let (L^X,δ) be an L-ts. $\mathcal{A}\subset\delta$ is called a crisp open cover of (L^X,δ) if, $\mathcal{A}$ is a cover of (L^X,δ) and for every $x\in X$ there exists $A\in\mathcal{A}$ such that $A(x)=1$. (L^X,δ) is called crisp under open covers (in short cuoc) if, every open cover of (L^X,δ) is a crisp open cover.

Proposition 4.12. If (L^X,δ) is a compact L-ts and $1\in M(L)$, then (L^X,δ) is cuoc.

Proof. Suppose (L^X,δ) is compact and $1\in M(L)$. Let $\mathcal{A}$ be any open cover of (L^X,δ). By compactness $\mathcal{A}$ has a finite subcover, say $\mathcal{B}$. Let $x\in X$. Then $1=\bigvee\{A(x):A\in\mathcal{B}\}$, and as $1\in M(L)$, there exists $A\in\mathcal{B}\subset\mathcal{A}$ such that $A(x)=1$. This ends the proof that (L^X,δ) is cuoc.

Even when $1\in M(L)$, cuoc L-ts' are not compact in general. As an example, take X to be any infinite ordinary set, L be any F-lattice with $1\in M(L)$, then (L^X,τ^{sdisc}) is cuoc but not compact.

The following example shows that the condition '$1\in M(L)$' in Proposition 4.12 cannot be dropped:

Example 4.13. Let L be the diamond-type lattice, i.e. $L=\{0,a,b,1\}$, where $0<a<1$, $0<b<1$, $a\perp b$, $0'=1$, $1'=0$, $a'=b$, $b'=a$. Take X be a doubleton $X=\{x,y\}$, $\delta=L^X$. Since L^X is finite, then (L^X,δ) is compact. On the other hand, since $\{x_a,x_b,y_a,y_b\}$ is an open cover of (L^X,δ) which is not a crisp open cover, then (L^X,δ) is not cuoc.

Theorem 4.14. Let (L^X,δ) be an L-ts. If (L^X,δ) is compact and $(L^X,\delta_{\mathcal{N}})$ is cuoc, then (L^X,δ) is $\mathcal{N}$-compact.

Proof. Let $\mathcal{A}$ be an $\mathcal{N}$-open cover of (L^X,δ) with $\mathcal{A}\subset\delta_{cof}$, say $\mathcal{A}=\{A_\alpha\wedge\mathcal{X}_{X-F_\alpha}:\alpha\in\Delta\}$ where $A_\alpha\in\delta$ and F_α is a finite subset of X for every $\alpha\in\Delta$. Then $\{A_\alpha:\alpha\in\Delta\}$ is an open cover of (L^X,δ). Since (L^X,δ) is compact, there exist $\alpha_1,\alpha_2,...,\alpha_n\in\Delta$ such that $\bigvee\{A_{\alpha_i}:1\leq i\leq n\}=1_X$. Put $B=\bigcup_{i=1}^{n}F_{\alpha_i}$. Since $(L^X,\delta_{\mathcal{N}})$ is cuoc, for every $b\in B$, we can choose $\alpha(b)\in\Delta$ such that $(A_{\alpha(b)}\wedge\mathcal{X}_{X-F_{\alpha(b)}})(b)=1$. Therefore, $\{A_{\alpha_i}\wedge\mathcal{X}_{X-F_{\alpha_i}}:1\leq i\leq n\}\cup\{A_{\alpha(b)}\wedge\mathcal{X}_{X-F_{\alpha(b)}}:b\in B\}$ is a finite subcover of $\mathcal{A}$. It follows that (L^X,δ) is $\mathcal{N}$-compact.

An I-ts (I^X,δ) is called Lindelöf (Wong, 1973), if every open cover of (I^X,δ) has a countable subcover.

The following definition generalizes Wong's Lindelöfness to include general L-ts'. It also uses $\mathcal{D}$-open L-sets to define another stronger type of Lindelöfness in L-ts'.

Definition 4.15. An L-ts (L^X,δ) is called Lindelöf (resp. $\mathcal{D}$-Lindelöf) if each open (resp. $\mathcal{D}$-open) cover of (L^X,δ) has a countable subcover.

Definition 4.16. An F-lattice L is called Lindelöf if, for each $A\subset L$ with $\bigvee A=1$, there exists a countable set $B\subset A$ such that $\bigvee B=1$.

The F-lattice I is Lindelöf. As an example of an F-lattice that is not Lindelöf, let L be the power set of $\mathbb{R}$, equip it with the inclusion order, then L is an F-lattice. However, as $A=\{\{x\}:x\in\mathbb{R}\}\subset L$, but there is not $B\subset A$ such that $\bigvee B=1$ and $B\neq A$, it follows that L is not Lindelöf.

For the case L is Lindelöf, by means of ω-open and $\mathcal{N}$-open L-sets, the next main result gives two characterizations of Lindelöf L-ts'.

Theorem 4.17. Let (L^X,δ) be an L-ts where L is Lindelöf. Then the following conditions are equivalent:

(i) (L^X,δ) is Lindelöf.

(ii) Each ω-open cover of (L^X,δ) has a countable subcover.

(iii) Each $\mathcal{N}$-open cover of (L^X,δ) has a countable subcover.

Proof. (i) $\Rightarrow$ (ii) Suppose that (L^X,δ) is Lindelöf. It is sufficient to see that each open cover $\mathcal{A}\subset\delta_{coc}$ has a countable subcover. Let $\mathcal{A}$ be an ω-open cover of (L^X,δ)

with $\mathcal{A} \subset \delta_{coc}$, say $\mathcal{A} = \{A_\alpha \wedge \mathcal{X}_{X-C_\alpha} : \alpha \in \Delta\}$ where $A_\alpha \in \delta$ and C_α is a countable subset of X for every $\alpha \in \Delta$. Then $\{A_\alpha : \alpha \in \Delta\}$ is an open cover of (L^X, δ). Since (L^X, δ) is Lindelöf, there exists a countable set $\Delta_\circ \subseteq \Delta$ such that $\{A_\alpha : \alpha \in \Delta_\circ\}$ covers 1_X. Put $B = \bigcup_{\alpha \in \Delta_\circ} C_\alpha$. Then B is countable. For every $b \in B$, $\bigvee \{(A_\alpha \wedge \mathcal{X}_{X-C_\alpha})(b) : \alpha \in \Delta\} = 1$ and as L is Lindelöf, there exists a countable set $\Delta_b \subseteq \Delta$ such that $\bigvee \{(A_\alpha \wedge \mathcal{X}_{A_\alpha})(b) : \alpha \in \Delta_b\} = 1$. Therefore, $\left\{A_\alpha \wedge \mathcal{X}_{X-C_\alpha} : \alpha \in \Delta_\circ \cup \left(\bigcup_{b \in B} \Delta_b\right)\right\}$ is a countable subcover of $\mathcal{A}$.

(ii) ⇒ (iii) Follows because $\delta_{\mathcal{N}} \subset \delta_\omega$.

(iii) ⇒ (i) Follows because $\delta \subset \delta_{\mathcal{N}}$.

Corollary 4.18. Let (I^X, δ) be an I-ts. Then the following conditions are equivalent:

(i) (I^X, δ) is Lindelöf.

(ii) Each ω-open cover of (I^X, δ) has a countable subcover.

(iii) Each $\mathcal{N}$-open cover of (I^X, δ) has a countable subcover.

Problem 4.19. Is it true that the condition 'L is Lindelöf', in Theorem 4.17 can be dropped?

Remark 4.20. It is clear that every $\mathcal{D}$-Lindelöf L-ts is ω-Lindelöf. However, it is not difficult to check that $(I^{\mathbb{R}}, \tau^{scoc})$ is an ω-Lindelöf L-ts which is not $\mathcal{D}$-Lindelöf.

Theorem 4.21. If (L^X, δ) is a $\mathcal{D}$-Lindelöf L-ts, then X countable.

Proof. Since $\{\mathcal{X}_{\{x\}} : x \in X\}$ is a $\mathcal{D}$-open cover of (L^X, δ), then there exists a countable set $D \subseteq X$ such that $\{\mathcal{X}_{\{d\}} : d \in D\}$ is a cover of 1_X. Thus, $X = D$. Hence X is countable.

The implication in Theorem 4.21 is not reversible, as the following example shows:

Example 4.22. Take $X = \{x\}$, L the power set of $\mathbb{R}$ equipped with the inclusion order, and $\delta = L^X$. Then $\{x_{\{r\}} : r \in \mathbb{R}\}$ is an open cover of (L^X, δ) and has no countable subcover, hence (L^X, δ) is even not Lindelöf.

The condition 'L is Lindelöf' is sufficient for the implication in Theorem 4.21 to be reversible:

Theorem 4.23. Let (L^X, δ) be an L-ts where L is Lindelöf. Then (L^X, δ) is $\mathcal{D}$-Lindelöf if and only if X is countable.

Proof. *Necessity*. Theorem 4.21.

Sufficiency. Let $\mathcal{A}$ be an open cover of $(L^X, \delta_{\mathcal{D}})$. For every $x \in X$, $\bigvee \{A(x) : A \in \mathcal{A}\} = 1$, and since L is Lindelöf, there exists a countable subfamily $\mathcal{A}_x$ of $\mathcal{A}$ such that $\bigvee \{A(x) : A \in \mathcal{A}_x\} = 1$. Thus, $\bigcup_{x \in X} \mathcal{A}_x$ is a countable subcover of $\mathcal{A}$, and hence (L^X, δ) is $\mathcal{D}$-Lindelöf.

As three L-topological properties, we know that each of compactness and $\mathcal{D}$-Lindelöfness is stronger than Lindelöfness. The question about relationships between compactness and $\mathcal{D}$-Lindelöfness is natural. The L-ts in Example 4.7 with the restriction that X is uncountable is compact and by Theorem 4.21, it is not $\mathcal{D}$-Lindelöf. The following is an example of a $\mathcal{D}$-Lindelöf L-ts that is not compact:

Example 4.24. By Theorem 4.23, the I-ts $(I^{\mathbb{N}}, \tau^{sdisc})$, where $\mathbb{N}$ is the set of natural numbers, is $\mathcal{D}$-Lindelöf, however, it is clear that it is not compact.

5. Conclusion

We introduced and investigated ω-open, $\mathcal{N}$-open, and $\mathcal{D}$-open as three weaker notions of open L-sets in L-ts' where L is an F-lattice. Then we used these notions to introduce several types of Chang's compactness and Wong's Lindelöfness. We introduced many results related to these types and we raised a question in Problem 4.19. By means of ω-open (resp. $\mathcal{N}$-open, $\mathcal{D}$-open) L-sets in L-ts', we hope to point out that a continuation of this paper should deal with reasonable modifications of α -compactness (Aygün, 2000), fuzzy compactness (Wong, 1973), paracompactness and Lindelöfness (Liu & Luo, 1997) and others.

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ثلاثة مفاهيم جديدة أضعف لمجموعات مفتوحة مشوشة و مفاهيم تغطية متعلقة

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خلاصة

نقول عن مجموعة جزئية A لفضاء طبولوجي إعتيادي (X, T) بأنها ω – مفتوحة ($\mathcal{N}$– مفتوحة) ، اذا كان لكل $x \in A$، يوجد $U \in T$ بحيث $x \in U$، و $U - A$ قابلة لعد (منتهية). نقوم في هذا البحث بتوسيع مفهومي ω – مفتوحة ($\mathcal{N}$ – مفتوحة إلى فضاءات L الطبولوجية حيث L هي F– شبكية، كما نقدم لأول مرة مفهوماً ثالثاً لمجموعات l أضعف من كل منهما. إذا كان لدينا فضاء L طبولوجي، تعطينا المفاهيم الجديدة ثلاثة فضاءات l طبولوجية أكثر دقة ، أنواع مختلفة من تراص شانغ، وكذلك سندخل مفهوم وونغ – ليندولوف نقوم بمقارنات بين المفاهيم الجديدة، و كذلك بين تلك الجديدة و المفاهيم المتعلقة بها. ثم نعطي تمييزاً للمفاهيم الجديدة ، وكذلك تمييزين إثنين لمفهوم وونغ – ليندولوف.