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Solutions around a regular α singular point of a sequential conformable fractional differential equation

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Abstract

In this work, firstly, some concepts of conformable fractional calculus in literature are given. Secondly, definitions of α-analytic point, α-ordinary point and regular α singular point are presented. Finally, the fractional power series solutions are given around a regular-singular point, in the case of variable coefficients for homogeneous sequential linear conformable fractional differential equations of order 2α.

Keywords: Conformable fractional derivative; fractional power series; regular α singular point; sequential conformable fractional differential equation; series solutions.

1. Introduction

Though fractional derivative idea is more than 300 years old, intensive studies about fractional calculus were carried out by many researches in the last and present centuries. Several mathematicians such as Liouville, Riemann, Weyl, Fourier, Abel, Leibniz, Grunwald and Letnikov made major contributions to the theory of fractional calculus. The most popular ones of fractional derivative definitions are Grunwald-Letnikov, Riemann-Liouville and Caputo definitions. For Grunwald-Letnikov, Riemann-Liouville, Caputo and other definitions and the characteristics of these definitions, we refer the reader to Kilbas *et al*. (2006); Miller (1993); Podlubny (1999).

(I) Grunwald-Letnikov definition:

$${}_aD_x^{\alpha} f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{j\to 0}^{\frac{x-a}{h}} (-1)^j \binom{\alpha}{j} f(x-jh)$$

(II) Riemann Liouville definition:

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\alpha-1} f(t)dt\,,$$
$$n-1<\alpha\le n$$

(III) Caputo definition:

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt\,,$$
$$n-1<\alpha\le n$$

By the help of these fractional derivative definitions, a lot of works have been made with regard to finding analytical approximate solutions (Shawagfeh, 2002; Das, 2009; Garg & Manohar, 2013) and exact solutions (Luchko & Srivastava, 1995; Gepreel & Omran 2012; Ghany & Hyder, 2014) of fractional ordinary and partial differential equations.

Recently, Khalil *et al*. (2014) gave a new definition of fractional derivative and fractional integral. This new definition based on a limit form as in usual derivatives. The authors also proved the product rule, the fractional Rolle's theorem and mean value theorem. This new theory is improved by Abdeljawad (2015). For instance, he gave Taylor power series representation and Laplace transform of certain functions, fractional integration by parts formulas, chain rule and Gronwall inequality.

In short time, a lot of studies about new fractional derivative definition have been presented. Some works in this field are with regard to conformable fractional fourier series (Khalil, 2014), the fractional power series solution of Legendre conformable fractional equation and Legendre fractional polynomials (Khalil & Abu Hammad, 2014), conformable fractional semigroups of operators (Horani *et al.*, 2014), conformable fractional calculus on arbitrary time scales (Benkhettou & Torres, 2015), fractional Newton mechanics (Chung, 2015), boundary value problems for conformable fractional differential equations (Batarfi *et al*., 2015; Anderson & Avery, 2015), existence and uniqueness theorems for sequential linear conformable fractional differential equations (Gökdoğan *et al*., 2015), the existence of solutions around an ordinary point of conformable fractional differential equation of order 2α (Ünal *et al*., 2015).

In this work, we analyze the existence of solutions around a regular α singular point of conformable fractional differential equation of order 2α.

2. Conformable fractional calculus

Definition 1. Let $f: [a, \infty) \to \mathbb{R}$ be given function. Then, the left conformable fractional derivative of f of order α is defined by

$$(T_\alpha^a f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon(x-a)^{1-\alpha}) - f(x)}{\varepsilon}$$

for all $x > a, \alpha \in (0,1]$. When $a = 0$, it is written as T_α. If $(T_\alpha f)(x)$ exists on (a,b) then $(T_\alpha^a f)(a) = \lim_{x \to a^+} (T_\alpha^a f)(x)$.

Definition 2. Let $f: [a, \infty) \to \mathbb{R}$ be given function. Then, the right conformable fractional derivative of f of order α is defined by

$$({}_\alpha^b T f)(x) = -\lim_{\varepsilon \to 0} \frac{f(x + \varepsilon(b-x)^{1-\alpha}) - f(x)}{\varepsilon}$$

for all $x < b, \alpha \in (0,1]$. If $({}_\alpha T f)(x)$ exists on (a,b) then $({}_\alpha^b T f)(b) = \lim_{x \to b^-} ({}_\alpha^b T f)(x)$.

Theorem 1. Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $x > 0$. Then,

(1) $\frac{d^\alpha}{dx^\alpha}(af + bg) = a\frac{d^\alpha f}{dx^\alpha} + b\frac{d^\alpha g}{dx^\alpha}$, for all $a, b \in \mathbb{R}$;

(2) $\frac{d^\alpha}{dx^\alpha}(x^p) = px^{p-\alpha}$, for all $p \in \mathbb{R}$;

(3) $\frac{d^\alpha}{dx^\alpha}(\lambda) = 0$, for all constant functions $f(x) = \lambda$;

(4) $\frac{d^\alpha}{dx^\alpha}(fg) = f\frac{d^\alpha}{dx^\alpha}(g) + g\frac{d^\alpha}{dx^\alpha}(f)$;

(5) $\frac{d^\alpha}{dx^\alpha}(f/g) = \frac{g\frac{d^\alpha}{dx^\alpha}(f) - f\frac{d^\alpha}{dx^\alpha}(g)}{g^2}$;

(6) If, in addition, f is differentiable, then, $\frac{d^\alpha}{dx^\alpha}(f(x)) = x^{1-\alpha}\frac{df}{dx}(x)$.

Theorem 2. Assume that f is infinitely α-differentiable function, for some $0 < \alpha \le 1$ at a neighborhood of a point x_0. Then, f has the fractional power series expansion:

$$f(x) = \sum_{k=0}^{\infty} \frac{\left({}^{(k)}T_\alpha^{x_0} f\right)(x_0)(x - x_0)^{k\alpha}}{\alpha^k k!},$$
$$x_0 < x < x_0 + R^{1/\alpha}, \qquad R > 0.$$

Here, $\left({}^{(k)}T_\alpha^{x_0} f\right)(x_0)$ means the application of the fractional derivative k times.

3. Conformable fractional differential equation and solutions around a regular α singular point

The most general sequential linear homogeneous (left) conformable fractional differential equation is

$$ {}^{(n)}T_\alpha^a y + a_{n-1}(x)\,{}^{(n-1)}T_\alpha^a y + \cdots + a_1(x) T_\alpha^a y + a_0(x) y = 0 \qquad (1)$$

where ${}^{(n)}T_\alpha^a y = T_\alpha^a T_\alpha^a \ldots T_\alpha^a y$, (n times).

Definition 3. Let $\alpha \in (0,1]$, $x_0 \in [a,b]$, $N(x_0)$ be a neighborhood of x_0 and $f(x)$ be a real function defined on $[a,b]$. In this case $f(x)$ is said to be α-analytic at x_0 if $f(x)$ can be expressed as a series of natural powers of $(x - x_0)^\alpha$ for all $x \in N(x_0)$. In other word, $f(x)$ can be expressed as following:

$$\sum_{k=0}^{\infty} c_k (x - x_0)^{k\alpha} (c_k \in R)$$

This series being definitely convergent for $|x - x_0| < \delta (\delta > 0)$. δ is the radius of convergence of the series.

Definition 4. Let $\alpha \in (0,1]$, $x_0 \in [a,b]$ and the functions $a_k(x)$ be α-analytic at $x_0 \in [a,b]$ for $k = 0,1,2,\ldots,n-1$. In this case, the point $x_0 \in [a,b]$ is said to be an α-ordinary point of the equation (1). If a point $x_0 \in [a,b]$ is not α-ordinary point, then it is said to be α singular.

Definition 5. Let $\alpha \in (0,1]$, $a_k(x)$ be α-singular at $x_0 \in [a,b]$ for $k = 0,1,2,\ldots,n-1$. If the functions $(x - x_0)^{(n-k)\alpha} a_k(x)$ are α-analytic at the point $x_0 \in [a,b]$ for $k = 0,1,2,\ldots,n-1$, then, the point x_0 is said to be a regular α singular point of (1). In the contrary case, x_0 is said to be an essential α singular point.

Example 1.a) We shall consider the following conformable fractional differential equations

$$x^\alpha T_\alpha y - y = 0,$$

$$x^{2\alpha}\,{}^2T_\alpha y - 2x^\alpha y = 0,$$

$$x^{2\alpha}\,{}^2T_\alpha y - 2x^\alpha T_\alpha y + x^{2\alpha} y = 0.$$

The point $x = 0$ is a regular α singular pointfor the above equations.

b)

$$(x-1)^{\alpha} T_{\alpha} y - y = 0,$$

$$(x-1)^{2\alpha}\, {}^{2}T_{\alpha} y - 2(x-1)^{\alpha} T_{\alpha} y + (x-1)^{2\alpha} y = 0.$$

For these equation, the point $x = 1$ is a regular α singular point.

Now, we consider the following homogeneous sequential linear fractional differential equation of order 2α:

$$(x-x_0)^{2\alpha} T_{\alpha}^{x_0} T_{\alpha}^{x_0} y + (x-x_0)^{\alpha} p(x) T_{\alpha}^{x_0} y + q(x) y = 0 \tag{2}$$

where $\alpha \in (0,1]$. If the point x_0 is a regular α singular point of the equation (2), then this point is α-analytic point for functions $p(x)$ and $q(x)$. In this case, functions $p(x)$ and $q(x)$, respectively, have the following series expansions:

$$p(x) = \sum_{k=0}^{\infty} p_k (x-x_0)^{k\alpha}\ (0 < x - x_0 < \delta_1;\ \delta_1 > 0)$$

and

$$q(x) = \sum_{k=0}^{\infty} q_k (x-x_0)^{k\alpha}\ (0 < x - x_0 < \delta_1;\ \delta_1 > 0).$$

Suppose that, for the equation (2), we have a solution the form

$$y(x;s) = \sum_{k=0}^{\infty} c_k(s)(x-x_0)^{(k+s)\alpha} \tag{3}$$

where let be $c_0 \neq 0$, s being a number to be determined.

If we substitute the equation (3) and conformable derivatives of the equation (3) in the equation (2), then we get

$$c_0 I_0(s)(x-x_0)^{s\alpha} + \sum_{k=1}^{\infty}\left[c_k I_0(k+s) + \sum_{j=0}^{k-1} c_j I_{k-j}(j+s)\right](x-x_0)^{(k+s)\alpha} = 0,$$

where

$$I_0(s) = \alpha^2 s(s-1) + \alpha s p_0 + q_0, \tag{4}$$

$$I_m(s) = p_m \alpha s + q_m \tag{5}$$

The equation (4) is called fractional indicial equation of the equation (2). The coefficients c_k is

$$c_k = -\frac{\sum_{j=0}^{k-1} a_j I_{k-j}(j+s)}{I_0(k+s)} \tag{6}$$

Theorem 3. Let $\alpha \in (0,1]$ and x_0 be a regular α singular point of the equation

$$(x-x_0)^{2\alpha} T_{\alpha}^{x_0} T_{\alpha}^{x_0} y + (x-x_0)^{\alpha} p(x) T_{\alpha}^{x_0} y + q(x) y = 0.$$

Let s_1, s_2 be distinct and $s_1 - s_2 \neq n$ for $n \in N$ two real roots of the fractional indicial equation. Then, there exist two linearly independent solution to the equation (2) as following:

$$y_1(x;s_1) = \sum_{k=0}^{\infty} c_k(s_1)(x-x_0)^{(k+s_1)\alpha} \tag{7}$$

$$y_2(x;s_2) = \sum_{k=0}^{\infty} c_k(s_2)(x-x_0)^{(k+s_2)\alpha} \tag{8}$$

for $x \in (x_0, x_0+\rho)$ with $\rho = \min\{\delta_1, \delta_2\}$ and initial conditions $c_0 = y(x_0)$, $\alpha c_1 = T_{\alpha} y(x_0)$. Since x_0 is a regular α singular point of the equation (2), by Definition 3 and Definition 5, it can be written that,

$$p(x) = \sum_{k=0}^{\infty} p_k (x-x_0)^{k\alpha},\ x \in [x_0, x_0+\delta_1];\ \delta_1 > 0 \tag{9}$$

and

$$q(x) = \sum_{k=0}^{\infty} q_k (x-x_0)^{k\alpha}, x \in [x_0, x_0+\delta_2];\ \delta_2 > 0. \tag{10}$$

Proof. We must prove that series the equation (3) converges for $\in (x_0, x_0+\rho)$. Let be $s = s_1$ and $s = s_2$ such that $s_1 - s_2$ is not a positive integer. We note that

$$I_0(s) = \alpha^2 (s-s_1)(s-s_2).$$

Hence, the following equations can be written,

$$I_0(s_1+k) = \alpha^2 k(k+s_1-s_2),$$

$$I_0(s_2+k) = \alpha^2 k(k+s_2-s_1).$$

Therefore, we get

$$I_0(s_1+k) \geq \alpha^2 k(k-|s_1-s_2|), \tag{11}$$

$$I_0(s_2+k) \geq \alpha^2 k(k-|s_2-s_1|) \tag{12}$$

Now, let r be any number such that $0 < r < \rho$. Series in (9) and (10) converge for $x \in [x_0, x_0 + r]$. Hence, there is a constant number $M > 0$ such that

$$|p_j| r^{j\alpha} \le M \qquad (j \in N), \tag{13}$$

$$|q_j| r^{j\alpha} \le M \qquad (j \in N). \tag{14}$$

Using (11), (12), (13) and (14) in (6), we have

$$|c_k(s_1)| \le \frac{M}{r^{k\alpha}} \frac{\sum_{j=0}^{k-1} \alpha(j+1+|s_1|) r^{j\alpha} |c_j(s_1)|}{\alpha^2 k(k - |s_1 - s_2|)}. \tag{15}$$

Now, let N be an integer number such that

$$N - 1 \le |s_1 - s_2| < N.$$

We define

$$C_0 = c_0(s_1) = 1, C_1 = |c_1(s_1)|, \dots, C_{N-1} = |c_{N-1}(s_1)|.$$

Let the coefficients C_k for $k \ge N$ be defined by

$$C_k = \frac{M}{r^{k\alpha}} \frac{\sum_{j=0}^{k-1} \alpha(j+1+|s_1|) r^{j\alpha} C_j(s_1)}{\alpha^2 k(k - |s_1 - s_2|)}. \tag{16}$$

From the definition of C_k and the equation (15), we see that

$$|c_k(s_1)| \le C_k \quad k = 0,1,2, \dots$$

We prove that the series

$$\sum_{k=0}^{\infty} C_k (x - x_0)^{k\alpha} \tag{17}$$

is convergent for $x \in (x_0, x_0 + \rho)$. By using (16), we obtain that

$$\begin{aligned} r^{\alpha} \alpha^2 (k+1)(k+1-|s_1 - s_2|) C_{k+1} &= \alpha^2 (k)(k - |s_1 - s_2|) C_k \\ &+ \alpha M (k+1+|s_1|) C_k. \end{aligned}$$

Hence,

$$\frac{C_{k+1}}{C_k} = \frac{\alpha^2 (k)(k - |s_1 - s_2|) + \alpha M (k+1+|s_1|)}{r^{\alpha} \alpha^2 (k+1)(k+1-|s_1 - s_2|)}$$

is obtained. By the help of the ratio test, we have that

$$\lim_{k\to\infty} \left| \frac{C_{k+1}(x - x_0)^{(k+1)\alpha}}{C_k (x - x_0)^{k\alpha}} \right| = \left(\frac{|x - x_0|}{r} \right)^{\alpha} < 1.$$

Thus, the series (17) converges for $x \in [x_0, x_0 + r]$. This implies that the series (3) converges for $x \in [x_0, x_0 + r]$ and s_1. Since r was any number satisfying $0 < r < \rho$, the series (3) converges for $x \in (x_0, x_0 + \rho)$.

Similarly, the same computations with s_1 replaced by s_2 everywhere show that the series (3) converges for $x \in (x_0, x_0 + \rho)$ and s_2.

Example 2. We consider the following conformable fractional differential equations:

$$x^{\alpha} T_{\alpha} T_{\alpha} y - \frac{1}{2} \alpha T_{\alpha} y + y = 0. \tag{18}$$

Here, $p(x) = -\frac{1}{2}\alpha, q(x) = x^{\alpha}$ and $x = 0$ is regular α singular point for the above equation. Roots of the fractional indicial equation for this equation are $s_1 = \frac{3}{2}$ and $s_2 = 0$. Hence, according to Theorem 3, this equation have solutions which forms of (7) and (8). That is,

$$y_1(x) = \sum_{k=0}^{\infty} c_k x^{\left(k+\frac{3}{2}\right)\alpha} \tag{19}$$

and

$$y_2(x) = \sum_{k=0}^{\infty} d_k x^{k\alpha}. \tag{20}$$

Substituting conformable fractional derivatives of (19) in (18),

$$c_k = -\frac{c_{k-1}}{\alpha^2 k\left(k+\frac{3}{2}\right)}, \ k = 1,2, \dots$$

are obtained. Similarly, for (20), we have

$$d_k = -\frac{d_{k-1}}{\alpha^2 k\left(k-\frac{3}{2}\right)}, k = 1,2, \dots$$

Hence, the first solution and the second solution of the equation (18) are obtained as, respectively,

$$y_1(x) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(5/2)}{\alpha^{2k} k! \, \Gamma(5/2 + k)} x^{\left(k+\frac{3}{2}\right)\alpha},$$

$$y_2(x) = d_0 \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(-1/2)}{\alpha^{2k} k! \, \Gamma(-1/2 + k)} x^{k\alpha},$$

Because of $\delta_1 \to \infty$ and $\delta_2 \to \infty$, it is that $\rho = \min \{\delta_1, \delta_2\} \to \infty$.

Theorem 4. Let x_0 be a regular α-singular point of the equation (2). For this equation, $p(x)$ and $q(x)$ have fractional power series expansion, for $x \in (x_0, x_0 + \rho)$ with $\rho > 0$. Let s_1 and s_2 be two real roots of the fractional indicial the equation (4).

If $s_1 = s_2$, then, there are two linearly independent solution and these solutions have, respectively, the following forms:

$$y_1(x;s_1) = \sum_{k=0}^{\infty} c_k (x-x_0)^{(k+s_1)\alpha} \; (c_0 \neq 0),$$

$$y_2(x;s_1) = ln|x-x_0| y_1(x;s_1) + \sum_{k=0}^{\infty} b_k (x-x_0)^{(k+s_1+1)\alpha}$$

for $x \in (x_0, x_0+\rho)$.

If $s_1 - s_2$ is a positive integer, then, linearly independent solutions have, respectively, the following forms:

$$y_1(x;s_1) = \sum_{k=0}^{\infty} c_k (x-x_0)^{(k+s_1)\alpha} \; (c_0 \neq 0),$$

$$y_2(x;s_2) = C.\,ln|x-x_0| y_1(x;s_1) + \sum_{k=0}^{\infty} b_k (x-x_0)^{(k+s_2)\alpha}$$

for $x \in (x_0, x_0+\rho)$ where C is a constant and it may happen zero.

Proof. For $s_1 \geq s_2$, according to Theorem 3, first solution of (2) has form

$$y_1(x;s_1) = \sum_{k=0}^{\infty} c_k(s_1)(x-x_0)^{(k+s_1)\alpha}. \tag{21}$$

We rewrite the equation (2) as following

$$T_\alpha^{x_0} T_\alpha^{x_0} y + P(x) T_\alpha^{x_0} y + Q(x) y = 0$$

where $P(x) = \frac{p(x)}{(x-x_0)^\alpha}$, $Q(x) = \frac{q(x)}{(x-x_0)^{2\alpha}}$.

By the help of conformable Abel's formulas in Abu Hammad & Khalil (2014), we write

$$y_2(x) = y_1(x) I_\alpha^{x_0}\left(\frac{e^{-I_\alpha^{x_0}(P(x))}}{[y_1(x)]^2}\right). \tag{22}$$

Now, let $s_1 - s_2 = N$ be such that N is non-negative integer. Hence, s_1 and $s_2 = s_1 - N$ are roots of indicial equation. Therefore, we get

$$-p_0 - 2\alpha s_1 = \alpha(-1-N), \tag{23}$$

$$P(x) = \frac{p_0 + p_1(x-x_0)^\alpha + p_2(x-x_0)^{2\alpha} + \cdots}{(x-x_0)^\alpha} = \frac{p_0}{(x-x_0)^\alpha} + p_1 + p_2(x-x_0)^\alpha + \cdots .$$

Hence,

$$e^{\left(-I_\alpha^{x_0}(P(x))\right)} = e^{\left(-I_\alpha^{x_0}\left(\frac{p_0}{(x-x_0)^\alpha} + p_1 + p_2(x-x_0)^\alpha + \cdots\right)\right)} = e^{\left(-p_0 ln|x-x_0| - \frac{p_1}{\alpha}(x-x_0)^\alpha - \frac{p_2}{2\alpha}(x-x_0)^{2\alpha} - \cdots\right)} = (x-x_0)^{-p_0} e^{\left(-\frac{p_1}{\alpha}(x-x_0)^\alpha - \frac{p_2}{2\alpha}(x-x_0)^{2\alpha} - \cdots\right)}.$$

That is,

$$e^{\left(-I_\alpha^{x_0}(P(x))\right)} = (x-x_0)^{-p_0}(1 + A_1(x-x_0)^\alpha + A_2(x-x_0)^{2\alpha} + \cdots). \tag{24}$$

Now, we choose $c_0 = 1$ and substitute (21) and (24) in (22). In this case, we get

$$y_2(x) = y_1(x) I_\alpha^{x_0}\left(\frac{(x-x_0)^{-p_0}(1 + A_1(x-x_0)^\alpha + A_2(x-x_0)^{2\alpha} + \cdots)}{(x-x_0)^{2s_1\alpha}(1 + c_1(x-x_0)^\alpha + c_2(x-x_0)^{2\alpha} + \cdots)^2}\right),$$

$$y_2(x) = y_1(x) I_\alpha^{x_0}\left(\frac{(x-x_0)^{-p_0-2s_1\alpha}(1 + A_1(x-x_0)^\alpha + A_2(x-x_0)^{2\alpha} + \cdots)}{(1 + B_1(x-x_0)^\alpha + B_2(x-x_0)^{2\alpha} + \cdots)}\right) = y_1(x) I_\alpha^{x_0}\left((x-x_0)^{\alpha(-1-N)}(1 + C_1(x-x_0)^\alpha + C_2(x-x_0)^{2\alpha} + \cdots)\right).$$

For $N = 0$, that is $s_1 = s_2$, we have

$$y_2(x) = y_1(x) I_\alpha^{x_0}((x-x_0)^{-\alpha} + C_1 + C_2(x-x_0)^\alpha + \cdots),$$

$$y_2(x) = y_1(x)\ln(x-x_0) + y_1(x)\left(\frac{C_1}{\alpha}(x-x_0)^\alpha + \frac{C_2}{2\alpha}(x-x_0)^{2\alpha} + \cdots\right),$$

$$y_2(x) = y_1(x)\ln(x-x_0) + (x-x_0)^{s_1\alpha}(1 + c_1(x-x_0)^\alpha + \cdots)\left(\frac{C_1}{\alpha}(x-x_0)^\alpha + \frac{C_2}{2\alpha}(x-x_0)^{2\alpha} + \cdots\right),$$

$$y_2(x) = y_1(x)\ln(x-x_0) + (x-x_0)^{s_1\alpha}(b_0(x-x_0)^\alpha + b_1(x-x_0)^{2\alpha} + b_3(x-x_0)^{3\alpha} \ldots).$$

Consequently, for $s_1 = s_2$, the general form of second solution is

$$y_2(x) = y_1(x) ln(x - x_0) + (x - x_0)^{(s_1+1)\alpha} \sum_{k=0}^{\infty} b_k (x - x_0)^{k\alpha}.$$

For $N > 0$, that is $s_1 - s_2 = N$, we have

$$y_2(x) = y_1(x) I_\alpha^{x_0} \Big((x - x_0)^{\alpha(-1-N)} (1 + C_1 (x - x_0)^\alpha + C_2 (x - x_0)^{2\alpha} + \cdots + C_N (x - x_0)^{N\alpha} + \cdots) \Big),$$

$$y_2(x) = y_1(x) I_\alpha^{x_0} \left(\left(\frac{C_N}{(x - x_0)^\alpha} + \frac{1}{(x - x_0)^{(1+N)\alpha}} + \frac{C_1}{(x - x_0)^{N\alpha}} + \cdots \right) \right),$$

$$y_2(x) = C_N y_1(x) ln(x - x_0) + y_1(x) \left(\frac{(x - x_0)^{-N\alpha}}{-N\alpha} + \frac{C_1 (x - x_0)^{(-N+1)\alpha}}{(-N+1)\alpha} + \cdots \right),$$

$$y_2(x) = C_N y_1(x) ln(x - x_0) + (x - x_0)^{(s_2+N)\alpha} (1 + c_1 (x - x_0)^\alpha + \cdots)(x - x_0)^{-N\alpha} \left(-\frac{1}{N\alpha} + \frac{C_1 (x - x_0)^\alpha}{(-N+1)\alpha} + \cdots \right).$$

Hence, for $N > 0$, the general form of second solution is

$$y_2(x) = C_N y_1(x) ln(x - x_0) + (x - x_0)^{s_2\alpha} \sum_{k=0}^{\infty} b_k (x - x_0)^{k\alpha}$$

where $b_0 = -\frac{c_0}{N} \neq 0$.

4. Conclusion

In this work, power series solutions around a regular α singular point in homogeneous sequential linear differential equation of conformable fractional of order 2α with variable coefficients are given. Firstly, definitions of α-ordinary point and regular α singular point is presented. Then, for distinct roots of indicial equation,where the differences between them is not positive integer, general form of solutions is given. Finally, for equal roots, distinct roots and roots in which the differences between them is integer, general form of solutions is obtained. It is appears that the results obtained in this work correspond to results, which are obtained in ordinary case.

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حلول حول نقطة منفردة نظامية لمعادلة تفاضلية كسرية مطاوعة

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خلاصة

في هذا البحث، نقوم أولاً بعرض بعض مفاهيم الحسبان الكسري المطاوع من المادة المنشورة. نقوم ثانياً بتعريف النقطة التحليلية من النوع α، النقطة الاعتيادية من النوع α، النقطة المنفردة النظامية من النوع α، أخيراً نعطي حلول متسلسلة قوى حول نقطة نظامية منفردة، في حالة المعاملات المتغيرة لمعادلات تفاضلية كسرية خطية مطاوعة متتالية و متجانسة من المرتبة 2α.