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Mannheim B-Curves in the Euclidean 3-Space E^3

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Abstract

In 1878, Mannheim defined the notion of a Mannheim pair of curves and in 2008 Liu and Wang gave the necessary and sufficient condition for the Mannheim curves using the Frenet frame as a key ingredient. In this article we consider Bishop frame and introduce Mannheim B-pair in the Euclidean 3-space E^3. Then a thorough examination of the relationships between curvatures of a Mannheim B-pair is given. As the results of this examination, the relations between the Bishop vectors and Frenet vectors of these curves are found. Besides the relation between the Bishop curvatures of the Mannheim B-curves is obtained as $k_2^* = \dfrac{k_1}{1+\lambda k_1}$.

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1. Introduction

At the corresponding points of associated curves, one of the Frenet vectors of a curve coincides with one of the Frenet vectors of other curve. This has attracted the attention of many mathematicians. One of the well-known curves is the Mannheim curve, where the principal normal line of a curve coincides with the binormal line of another curve at the corresponding points of these curves. The first study of Mannheim curves has been presented by Mannheim in 1878 and has a special position in the theory of curves (Blum, 1966). Other studies have been revealed, which introduce some characterized properties in the Euclidean and Minkowski space (Lee, 2011; Liu & Wang, 2008; Orbay & Kasap, 2009; Öztekin & Ergüt, 2011). The generalizations of the Mannheim curves in the 4-dimensional spaces have been given (Matsuda & Yorozu, 2009; Akyiğit, *et al.* 2011). Later, Mannheim offset the ruled surfaces and dual Mannheim curves have been defined in Orbay *et al.* 2009; Özkaldı *et al.* 2009; Güngör & Tosun, 2010). Apart from these, some properties of Mannheim curves have been analyzed according to different frames such as the weakened Mannheim curves, quaternionic Mannheim curves and quaternionic Mannheim curves of $Aw(k)-$ type (Karacan, 2011; Okuyucu, 2013; Önder & Kızıltuğ, 2012; Kızıltuğ & Yaylı, 2015).

In this work, Mannheim curves have been introduced according to the Bishop frame and the relationships between Frenet, Bishop vectors and curvatures of these curves with respect to each other in the Euclidean 3-space E^3 have been obtained.

2. Preliminaries

In this section some fundamental characteristics of space curves has been discussed from the point of the differential geometry in three dimensional Euclidean space.

Let $\{\mathrm{T},\mathrm{N},\mathrm{B},\kappa,\tau\}$ and $\{T,N_1,N_2,k_1,k_2\}$ be the Frenet and Bishop apparatus of regular curve C with the arc-length parameter s respectively. If we denote the differentials according to the arc-length parameter then the Frenet and Bishop formulas of the curve C are expressed as (Serret, 1851; Bishop, 1975).

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N, \end{aligned} \tag{1}$$

and

$$\begin{aligned} T' &= k_1 N_1 + k_2 N_2, \\ N_1{}' &= -k_1 T, \\ N_2{}' &= -k_2 T. \end{aligned} \tag{2}$$

Also, the relations between Frenet and Bishop vectors and the curvatures of Frenet and Bishop frames are given as follows:

$$\begin{aligned} T &= C', \\ N &= \cos\theta N_1 + \sin\theta N_2, \\ B &= -\sin\theta N_1 + \cos\theta N_2, \end{aligned} \tag{3}$$

and

$$\tau(s) = -\theta'(s),\ \kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)} \tag{4}$$

where $\theta(s) = \arctan\dfrac{k_2(s)}{k_1(s)}$, $N_2 = T \Lambda N_1$.

Furthermore, the relations

$$\begin{aligned} k_1(s) &= \kappa(s)\cos\theta(s), \\ k_2(s) &= \kappa(s)\sin\theta(s), \end{aligned} \tag{5}$$

can be written for the Bishop curvatures of the curve C (Orbay & Kasap, 2009).

If there exists one to one correspondence between the points of the space curves C and C^* such that the binormal vector of C is in the direction of the principal normal vector of the curve C^*, then the (C, C^*) curve couple is called Mannheim pairs (Liu & Wang, 2008).

The space curve C is a Mannheim curve if and only if the curvature κ and the torsion τ of C satisfy the following equation

$$\kappa(s) = \lambda\left(\kappa^2(s) + \tau^2(s)\right)$$

where λ is a non-zero constant, (Blum, 1966).

If a (C, C^*) curve couple is a Mannheim pair in E^3, then the torsion of the curve C^* is $\tau^* = \dfrac{\kappa}{\lambda\tau}$, (Orbay & Kasap, 2009).

Moreover, a (C, C^*) curve couple is a Mannheim pair if and only if

$$\dot{\tau} = \frac{\kappa}{\lambda}\left(1 + \lambda^2\tau^2\right)$$

where $\lambda \neq 0$ (Liu & Wang 2008).

3. Mannheim B-curves

Now, the Mannheim B-curves and some characterizations of these curves will be introduced.

Definition 1**.** Let C and C^* be unit speed curves with the arc-length parameters of s and s^* respectively. Denote the Bishop apparatus of C and C^* by $\{T, N_1, N_2, k_1, k_2\}$ and $\{T^*, N_1^*, N_2^*, k_1^*, k_2^*\}$ respectively. If the Bishop vector N_1 coincides with the Bishop vector N_2^* at the corresponding points of the curves C and C^* then the curve C is said to be a Mannheim partner B-curve of C^* or a (C, C^*) curve couple is called Mannheim B-pair, see Fig.1.

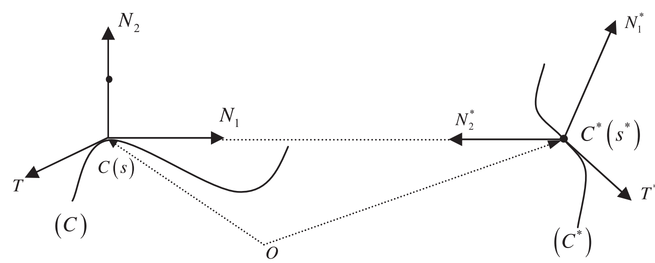

Fig. 1. Mannheim B-Curves

Let α be the angle between the tangents T and T^* of (C, C^*) Mannheim B-pair. Thus from the definition of Mannheim B-pair the following matrix representation can be written

$$\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \\ -\sin\alpha & \cos\alpha & 0 \end{bmatrix} \cdot \begin{bmatrix} T^* \\ N_1^* \\ N_2^* \end{bmatrix} \tag{6}$$

Theorem 1. The distance between the corresponding points of the Mannheim B-curves is constant in E^3.

Proof. Let us consider that the pair (C, C^*) is a Mannheim B-pair. Then we can write

$$C(s) = C^*(s^*) + \lambda(s^*)N_2^*(s^*) \tag{7}$$

where λ is a function of s^*, see Figure 1. If we take the derivative of the equation (7) with respect to s^* and apply the Bishop formulas, we obtain

$$T\frac{ds}{ds^*} = \left(1 - \lambda k_2^*\right)T^* + \lambda' N_2^*. \tag{8}$$

Since T is orthogonal to N_2^* we get $\lambda' = 0$. That is, λ is a non-zero constant.

On the other hand, from the distance function between two points, we have

$$d\left(C(s), C^*(s^*)\right) = \left\| C(s) - C^*(s^*) \right\| = \left\| \lambda N_2^* \right\| = |\lambda|.$$

Thus the distance between each corresponding points of the Mannheim B-pair is constant.

Theorem 2. Let (C, C^*) be a Mannheim B-pair in E^3. Then the relationships between the Bishop vectors of C and C^* are given as follows $T = \mu T^*,\ N_1 = N_2^*,\ N_2 = \mu N_1^*$ such that

$$\mu = \begin{cases} 1, & \text{for } \alpha = 0, \\ -1, & \text{for } \alpha = \pi\ . \end{cases}$$

where α is the angle between the tangent vectors of C and C^*

Proof. If (C, C^*) is a Mannheim B-pair in E^3 then using the equation (8) and Theorem 1, we obtain

$$T\frac{ds}{ds^*} = \left(1-\lambda k_2^*\right)T^*. \tag{9}$$

Since α is the angle between the tangent vectors T and T^* we get

$$\frac{ds}{ds^*}\cos\alpha = 1-\lambda k_2^*. \tag{10}$$

On the other hand, from the equations (6) and (9), it is easily seen that

$$\sin\alpha = 0.$$

So for the Bishop vectors of the pair (C, C^*) the equation (6) gives

$$T=\mu T^*,\quad N_1 = N_2^*,\quad N_2=\mu N_1^*,$$

where

$$\mu = \begin{cases} 1, & \text{for } \alpha = 0\ , \\ -1, & \text{for } \alpha=\pi\ . \end{cases}.$$

Theorem 3. Let (C, C^*) be a Mannheim B-pair and their Frenet apparatus be $\{T,N,B,\kappa,\tau\}$ and $\{T^*,N^*,B^*,\kappa^*,\tau^*\}$ in E^3, respectively. Then the relationships between the Frenet vectors of C and C^* are given by

$$\begin{aligned} T^* &= \mu T\ , \\ N^* &= \sin\left(\theta^*-\mu\theta\right)N - \mu\cos\left(\theta^*-\mu\theta\right)B, \\ B^* &= cos(\theta^*-\mu\theta)N - \mu sin(\theta^*-\mu\theta)B. \end{aligned}$$

where

$$\mu = \begin{cases} 1, & \text{for } \alpha = 0, \\ -1, & \text{for } \alpha=\pi. \end{cases}$$

Proof. Let the pair (C, C^*) be a Mannheim B-pair in E^3. By considering the equations (3), (6) and Theorem 2, we can write

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^* & \sin\theta^* \\ 0 & -\sin\theta^* & \cos\theta^* \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & 0 \\ 0 & 0 & -\cos\alpha \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where θ is the angle between the Frenet vector N and the Bishop vector N_1 of the curve C and also, θ^* is the angle between the Frenet vector N^* and the Bishop vector N_1^* of the curve C^*. Thus, if the necessary calculations are done we obtain

$$\begin{aligned} T^* &= \cos\alpha\, T, \\ N^* &= \left\{\sin\theta^*\cos\theta - \sin\theta\cos\alpha\cos\theta^*\right\}N \\ &\quad + \left\{-\sin\theta^*\sin\theta - \cos\theta\cos\alpha\cos\theta^*\right\}B, \\ B^* &= \left\{\cos\theta^*\cos\theta + \sin\theta\ \sin\theta^*\ \cos\alpha\right\}N \\ &\quad + \left\{-\cos\theta^*\sin\theta + \cos\theta\sin\theta^*\ \cos\alpha\right\}B. \end{aligned}$$

If we distinguish $\mu = 1$ or $\mu = -1$ for the values 0 or π of the angle α, respectively, then we have the relationships between the Frenet vectors of the curves C and C^* as follows;

$$\begin{aligned} T^* &= \mu T, \\ N^* &= sin(\theta^*-\mu\theta)N - \mu cos(\theta^*-\mu\theta)B, \\ B^* &= cos(\theta^*-\mu\theta)N + \mu sin(\theta^*-\mu\theta)B, \end{aligned}$$

Thus the next corollary can be given.

Corollary 1. Let (C, C^*) be a Mannheim B-pair in E^3. If the (C, C^*) curve couple is Mannheim pair then $\theta^* = \mu\theta$.

Theorem 4. Let (C, C^*) be a Mannheim B-pair in E^3. Then the relationship between the second Bishop curvature k_2^* of the curve C^* and the first Bishop curvature k_1 of the curve C is given as follows

$$k_2^* = \frac{k_1}{1+\lambda k_1}.$$

Proof. Let (C, C^*) be a Mannheim B-pair. Then from Theorem 2 we have

$$\cos\alpha = \left(1-\lambda k_2^*\right)\frac{ds^*}{ds} \quad \text{and} \quad \sin\alpha = 0.$$

Furthermore, from the equality $\sin^2\alpha + \cos^2\alpha = 1$ we find $\cos^2\alpha = 1$. This gives

$$\left|1-\lambda k_2^*\right| = \left|\frac{ds}{ds^*}\right|. \tag{11}$$

Differentiating the equation (4) with respect to s^* and considering the equation (2) we find

$$\begin{aligned} \left(k_1N_1 + k_2N_2\right)\left(\frac{ds}{ds^*}\right)^2 + T\,\frac{d^2s}{ds^{*2}} &= \left(-\lambda\, k_2^{*\prime}\right)T^* \\ &\quad + \left(1-\lambda\, k_2^*\right)k_1^*N_1^* \\ &\quad + \left(1-\lambda\, k_2^*\right)k_2^*N_2^* \end{aligned} \tag{12}$$

From the definition of the Mannheim B-pair, we can easily see that

$$k_1\left(\frac{ds}{ds^*}\right)^2 = \left(1-\lambda\, k_2^*\right)k_2^* \tag{13}$$

Substituting the equation (11) into the equation (13) and arranging this equation, we get

$$k_1 = \frac{k_2^*}{1-\lambda k_2^*} \text{ or } k_2^* = \frac{k_1}{1+\lambda k_1}.$$

Theorem 5. Let (C, C^*) be a Mannheim B-pair in E^3. Then the following equalities hold:

$$k_1 \frac{ds}{ds^*} = \cos\alpha\, k_2^*,$$

$$k_1^* + k_2 \frac{ds}{ds^*} = 0.$$

Proof. Suppose that (C, C^*) is a Mannheim B-pair. The equation (6) and Theorem 2 give us

$$T = \cos\alpha T^*. \tag{14}$$

Differentiating this equation with respect to s^* and considering (2) we find

$$\left(k_1 N_1 + k_2 N_2\right)\frac{ds}{ds^*} = \cos\alpha\, k_1^*\, N_1^* + \cos\alpha\, k_2^*\, N_2^*. \tag{15}$$

If we take the inner product both sides of the above equation with the vector N_1^* then it is seen that

$$-k_2 \frac{ds}{ds^*}\cos\alpha = \cos\alpha\, k_1^*.$$

Subsequently, we get

$$k_1^* + k_2 \frac{ds}{ds^*} = 0.$$

Similarly, if we take the inner product both sides of the equation (15) with the vector N_2^*, then we have

$$k_1 \frac{ds}{ds^*} = \cos\alpha\, k_2^*.$$

Thus, the following corollaries can be given from Theorem 4 and Theorem 5.

Corollary 2. If the couple (C, C^*) is a Mannheim B-pair then there is a relationship between the first Frenet curvatures of the curves C and C^* as follows;

$$\kappa^* = \kappa\left|\frac{ds}{ds^*}\right|.$$

Proof. Assume that the pair (C, C^*) is a Mannheim B-pair. Considering Theorem 4, Theorem 5 and the equation (11), the sum of the squares of the curvatures of the curve C is given by

$$k_1^2 + k_2^2 = \left(k_1^{*2} + k_2^{*2}\right)\left(\frac{ds^*}{ds}\right)^2.$$

Moreover, if we apply to the equation (5) then the last equation becomes

$$\kappa^* = \kappa\left|\frac{ds}{ds^*}\right|.$$

Corollary 3. Let (C, C^*) be a Mannheim B-pair in E^3. The ratio of the curvatures k_1 and k_2 is constant if and only if the ratio between the curvatures k_1^* and k_2^* is constant, too.

Proof. Assume that the pair (C, C^*) is a Mannheim B-pair in E^3. Then, Theorem 4 yields

$$\frac{k_1}{k_2} = -\frac{k_2^*}{k_1^*}\mu,$$

where

$$\mu = \begin{cases} 1, & \text{for } \alpha = 0, \\ -1, & \text{for } \alpha = \pi. \end{cases}$$

Then, it is obvious that $\frac{k_1}{k_2}$ is constant if and only if $\frac{k_1^*}{k_2^*}$ is constant.

Example 1. Let us consider the curve (C) given by the parametric equation

$$C(s) = \left(3\cos\frac{s}{5},\ 3\sin\frac{s}{5},\ \frac{4s}{5}\right).$$

The Frenet vectors of $C(s)$ are obtained as follows

$$T(s) = \left(-\frac{3}{5}\sin\frac{s}{5}, \frac{3}{5}\cos\frac{s}{5},\ \frac{4}{5}\right),$$

$$N(s) = \left(-\cos\frac{s}{5}, -\sin\frac{s}{5},\ 0\right),$$

$$B(s) = \left(\frac{4}{5}\sin\frac{s}{5}, -\frac{4}{5}\cos\frac{s}{5},\ \frac{3}{5}\right),$$

and the curvatures of this curve is

$$\kappa(s) = \frac{3}{25} \text{ and } \tau(s) = \frac{4}{25}.$$

The parametric representation of the Mannheim partner (with respect to Frenet frame) curve (C^{**}) of the curve (C) is obtained as

$$\begin{aligned} C^{**}(s) &= C(s) + \lambda B(s) \\ &= \left(3\cos\frac{s}{5} + \frac{4\lambda}{5}\sin\frac{s}{5}, 3\sin\frac{s}{5} - \frac{4\lambda}{5}\cos\frac{s}{5}, \frac{4s}{5} + \frac{3\lambda}{5}\right). \end{aligned}$$

The Bishop vectors of the curve (C) are found as

$$T(s)=\left(-\frac{3}{5}\sin\frac{s}{5},\frac{3}{5}\cos\frac{s}{5},\ \frac{4}{5}\right),$$

$$N_1(s)=\left(-\cos\theta(s)\cos\frac{s}{5}-\frac{4}{5}\sin\theta(s)\sin\frac{s}{5},\right.$$
$$\left.-\cos\theta(s)\sin\frac{s}{5}+\frac{4}{5}\sin\theta(s)\cos\frac{s}{5},\ -\frac{3}{5}\sin\theta(s)\right),$$

$$N_2(s)=\left(-\sin\theta(s)\cos\frac{s}{5}-\frac{4}{5}\cos\theta(s)\sin\frac{s}{5},\right.$$
$$\left.-\sin\theta(s)\sin\frac{s}{5}-\frac{4}{5}\cos\theta(s)\cos\frac{s}{5},\ \frac{3}{5}\cos\theta(s)\right)$$

Here it is easily seen that $\theta(s)=-\frac{4s}{25}$.

The parametric representation of the Mannheim partner (with respect to Bishop frame) B-curve (C^*) of the curve (C) is given by

$$C^*(s)=C(s)+\lambda N_1(s)$$
$$=\left(3\cos\frac{s}{5}-\lambda\left(\cos\frac{4s}{25}\cos\frac{s}{5}-\frac{4}{5}\sin\frac{4s}{25}\sin\frac{s}{5}\right),\right.$$
$$\left.3\sin\frac{s}{5}-\lambda\left(\cos\frac{4s}{25}\sin\frac{s}{5}+\frac{4}{5}\sin\frac{4s}{25}\cos\frac{s}{5}\right),\frac{4s}{5}+\frac{3\lambda}{5}\sin\frac{4s}{25}\right)$$

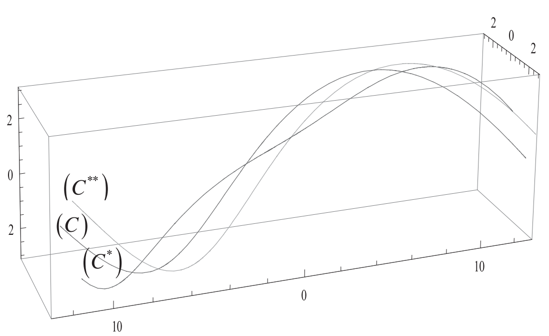

Fig. 2. Mannheim partner curve (C^{**}) and Mannheim partner B-curve (C^*) of the curve (C) for $\lambda=1$ and $-5\pi\le s\le 5\pi$. This figure has been drawn with the aid of the Mathematica ParametricPlot3D command.

4. Conclusion

In the present paper, the definition of Mannheim B-pair according to Bishop frame in the Euclidean 3-space E^3 has been provided which has been followed by the calculations of the relations between Bishop and Frenet vectors of Manheim B-pair. Besides, some theorems and results about the curvatures of Mannheim B-pair have been stated. In addition, the curve theory is a fundamental structure of differential geometry. So, this paper will attract the attention of the geometers. Thus, we hope that researchers who study in the area of biology and computer graphics can benefit from these results since the applications of Bishop frame are used in these areas.

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منحنيات B ما نهايم في الفضاء الأقليدي الثلاثي E^3

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خلاصة

عرف ما نهايم، في العام 1878، مفهوم زوج من المنحنيات الذي سمى باسمه و أعطى ليووانغ في العام 2008 الشرط اللازم و الكافي لمنحنيات ما نهايم و ذلك باستخدام إطار فرينية كمكون أساسي. نقوم في هذا البحث بدراسة إطار بيشوب و نعرف مفهوم زوج B ما نهايم في الفضاء الاقليدي الثلاثيE^3. ثم نقوم بدراسة معمقة للعلاقات بين تقوسات زوج B ما نهايم. كنتيجة لهذه الدراسة، نحصل على العلاقات بين متجهات بيشوب و متجهات فرينية لهذه المنحنيات. إضافة إلى ذلك، نحصل على العلاقة بين تقوسات بيشوب لمنحنيات B ما نهايم كالتالي: $k_2^* = \dfrac{k_1}{1+\lambda k_1}$