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Exact distribution of the sum of two correlated chi-square variables and its application

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ABSTRACT

The exact distribution of the sum of two chi-square random variables is known if the variables are independent. We derive the exact distribution of the sum of two correlated chi-square variables when they are correlated through a bivariate chi-square distribution. The distribution is important in estimating the common variance of a bivariate normal population. Some properties of the distribution, namely, the characteristic function, cumulative distribution function, raw moments, mean centered moments, coefficient of skewness and kurtosis are derived. The graph of the density function is also presented.

Keywords: Bivariate chi-square distribution; characteristic function; correlated chi-square variables; cumulative distribution function; sum of correlated chi-square variables.

INTRODUCTION

Let S_1^2 and S_2^2 be sample variances based on a sample of size $N = m + 1$ from a bivariate normal distribution with unknown means, unknown variances σ_1^2 and σ_2^2, and correlation coefficient ρ $(-1 < \rho < 1)$. The joint density function of $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$, called the bivariate chi-square distribution, follows from Krishnaiah *et al.* (1963). It was derived by Gunst & Webster (1973) by using Canonical Correlation Analysis. The product moment correlation coefficient between U and V can be calculated to be ρ^2. For the estimation of correlation coefficient by modern techniques, we refer to Ahmed (1992). In case the correlation coefficient (ρ) between the parent variables vanishes, the density function of U and V becomes that of the product of two independent chi-square variables each with m degrees of freedom.

The bivariate chi-square distribution plays an important role in radar systems, the detection of signals in noise etc. (Lawson & Uhlenbeck, 1950). It is also useful in determining the probability of missing a target by specified distance when firing projectiles or missiles.

Gerkmann & Martin (2009) has applied the results of correlated bivariate chi-square model of Joarder (2009) to derive explicit expression for the variance and covariance of correlated spectral amplitudes and the resulting cepstral coefficients. The results in their work allow for cepstral smoothing of spectral quantities without affecting their signal power. Interested readers may go through the paper and the references therein.

The exact density function of the linear combination of independent gamma variables has been derived by Provost (1988). A special case of his paper is the density function of the linear combination of independent chi-square variables. However, since the results in the correlated case are unknown, we consider studying the distribution of the sum of two correlated chi-square variables.

The distribution of the sum of correlated chi-square variables arises in the construction of confidence intervals for the common variance of multivariate normal populations with correlation structure having unknown correlation coefficients. By application of the inversion formula to the characteristic function of the sum of correlated chi-squares, Gordon & Ramig (1983) have derived an integral form of the cumulative distribution function (CDF) of the sum and used trapezoidal rule to approximate it. Thus the cumulative probabilities from their CDF are approximate values which depend on the type of numerical approximation technique one employs.

In Section 3, we derive the exact distribution of the sum of two correlated chi-square variables in terms of a generalized hypergeometric function. The graph of the density function of the sum is presented at the end of the paper. We also derive the exact cumulative distribution function (CDF) of the sum of two correlated chi-squares in terms of incomplete gamma function. In Section 4, we derive the characteristic function of the sum of two correlated chi-square variables. In Section 5, we derive some properties of the distribution of the sum, namely, raw moments, mean centered moments, coefficient of skewness and kurtosis. In Section 6, we also present an application of the distribution in the estimation of common variance of a bivariate normal distribution. If the results of the paper are specialized to the uncorrelated case ($\rho = 0$), then as expected, they match with that of the independent case.

SOME PRELIMINARIES

The duplication formula of gamma function is given by

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z + (1/2)). \tag{1}$$

We will be using the following product of k consecutive numbers:

$$a_{\{k\}} = a(a+1)\cdots(a+k-1),\ a^{\{k\}} = a(a-1)\cdots(a-k+1),\ \textit{with}\ a_{\{0\}} = 1\ \textit{and}\ a^{\{0\}} = 1, \quad (2)$$

where k is a non-negative integer.

The hypergeometric function ${}_pF_q(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; z)$ is defined by

$${}_pF_q(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_{\{k\}}(a_2)_{\{k\}} \cdots (a_p)_{\{k\}}}{(b_1)_{\{k\}}(b_2)_{\{k\}} \cdots (b_q)_{\{k\}}} \frac{z^k}{k!}, \quad (3)$$

(Gradshteyn & Ryzhik, 1994, p1065). The incomplete gamma function is defined by

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} e^{-t} dt, \quad (4)$$

where $Re(\alpha) > 0$ (Gradshteyn & Ryzhik, 1994, Equation 8.350, p949).

The joint density function of bivariate chi-square distribution (Gunst & Webster, 1973) is expressed below in terms of generalized hypergeometric function.

Theorem 2.1 Let S_1^2 and S_2^2 be variances for a sample drawn from a bivariate normal distribution as discussed in the introduction. Then the joint density function of $U = mS_1^2/\sigma_1^2$ and $V = mS_2^2/\sigma_2^2$ is given by the following function:

$$f_{U,V}(u, v) = \frac{(uv)^{(m-2)/2}}{2^m\, \Gamma^2(m/2)(1-\rho^2)^{m/2}} \exp\left(-\frac{u+v}{2-2\rho^2}\right) {}_0F_1\left(\frac{m}{2}; \frac{\rho^2 uv}{(2-2\rho^2)^2}\right), \quad (5)$$

where $m > 2$, $-1 < \rho < 1$ and ${}_0F_1(;b;z)$ is defined in Eq. 3.

The random variables U and V are said to have a correlated bivariate chi-square distribution each with m degrees of freedom, if its density function is given by Eq. 5. In case $\rho = 0$, it can be verified that the density function in Eq. 5, would be $f_U(u)f_V(v)$, the product of density functions of two independent chi-square random variables U and V each having m degrees of freedom.

THE DENSITY FUNCTION AND THE CDF OF THE SUM

In the following theorem, we derive the density function of the sum of two correlated chi-square variables.

Theorem 3.1 Let U and V be two correlated chi-square random variables with joint density function given by Eq. 5. Then the density function of $Z = U + V$ is given by

$$f_Z(z) = \frac{(1-\rho^2)^{-m/2}}{2^m\,\Gamma(m)} z^{m-1} \exp\left(\frac{-z}{2-2\rho^2}\right) {}_0F_1\left(\frac{m+1}{2};\frac{\rho^2 z^2}{(4-4\rho^2)^2}\right), 0 \le z < \infty, \quad (6)$$

where $m > 2, -1 < \rho < 1$ and ${}_0F_1(;b;z)$ is defined in Eq. 3.

Proof. Let $z = h_1(u,v) = u+v,\ w = h_2(u,v) = uv,\ u = h_1^{-1}(z,w), v = h_2^{-1}(z,w)$ in Eq. 5 so that the joint density function of Z and W is given by

$$f_{Z,W}(z,w) = g_1(h_1^{-1}(z,w), h_2^{-1}(z,w))|J_1| + g_2(h_1^{-1}(z,w), h_2^{-1}(z,w))|J_2|,$$

where g_1 and g_2 are the density functions of Z and W in the domain $D_1 = \{(u,v), u > v\}$ and $D_2 = \{(u,v) : u < v\}$ respectively and $|J_i|, (i = 1,2)$ is the Jacobian of transformation. In $D_1 = \{(u,v) : u > v\}$ we have $2u = z + \sqrt{z^2 - 4w}$, $2v = z - \sqrt{z^2 - 4w}$ so that

$$\frac{\partial u}{\partial y}\frac{\partial v}{\partial w} - \frac{\partial u}{\partial w}\frac{\partial v}{\partial y} = (y^2 - 4w)^{-1/2}$$

yielding $J(u, v \to z, w) = (z^2 - 4w)^{-1/2}$, $z > 2\sqrt{w}$. In $D_2 = \{(u,v) : u < v\}$, we have $2u = z - \sqrt{z^2-4w}$, $2v = z + \sqrt{z^2-4w}$ and as above, it can be proved that

$$\frac{\partial u}{\partial z}\frac{\partial v}{\partial w} - \frac{\partial u}{\partial w}\frac{\partial v}{\partial z} = -(z^2 - 4w)^{-1/2},$$

so that the Jacobian of the transformation is $|J(u, v \to z, w)| = (z^2 - 4w)^{-1/2}, z > 2\sqrt{w}$. Then from Eq. 5, the joint density function of $Z = U + V$ and $W = UV$ is given by

$$f_{Z,W}(z,w) \propto w^{(m/2)-1}\ (z^2 - 4w)^{-1/2} \exp\left(\frac{-z}{2-2\rho^2}\right) {}_0F_1\left(\frac{m}{2};\frac{\rho^2 w}{(2-2\rho^2)^2}\right),$$

where $w > 0, z > 2\sqrt{w},\ m > 2, -1 < \rho < 1$. By integrating the above joint density function over w, we have

$$f_Z(z) \propto \exp\left(\frac{-z}{2-2\rho^2}\right) I(z;m,\rho), \quad (7)$$

where $I(z;m,\rho) = \int_0^{z^2/4} w^{(m/2)-1}\ (z^2-4w)^{-1/2}\ {}_0F_1\left(\frac{m}{2};\frac{\rho^2 w}{(2-2\rho^2)^2}\right) dw.$

By expanding the hypergeometric function in the above integral and evaluating the resulting beta integral, we have

$$I(z;m,\rho) = \frac{z^{m-1}\Gamma(m/2)}{2^m}\ {}_0F_1\left(\frac{m+1}{2};\frac{\rho^2 z^2}{(4-4\rho^2)^2}\right), \quad 0 \le z < \infty.$$

Putting this in Eq. 7, we have

$$f_Z(z) \propto z^{m-1} \exp\left(\frac{-z}{2-2\rho^2}\right) {}_0F_1\left(\frac{m+1}{2}; \frac{\rho^2 z^2}{(4-4\rho^2)^2}\right).$$

The normalizing constant can be determined as what we have in Eq. 6 by integrating the above density over $0 \leq z < \infty$.

The density function in Eq. 6 of the sum of two correlated chi-square variables may be denoted by $\chi^2_{2m,\rho}$. Figure 1 below shows the graph of this density function in Eq. 6 for various values of ρ, and for $m = 5$. If $\rho = 0$, then the density function in Eq. 6 simplifies to that of a

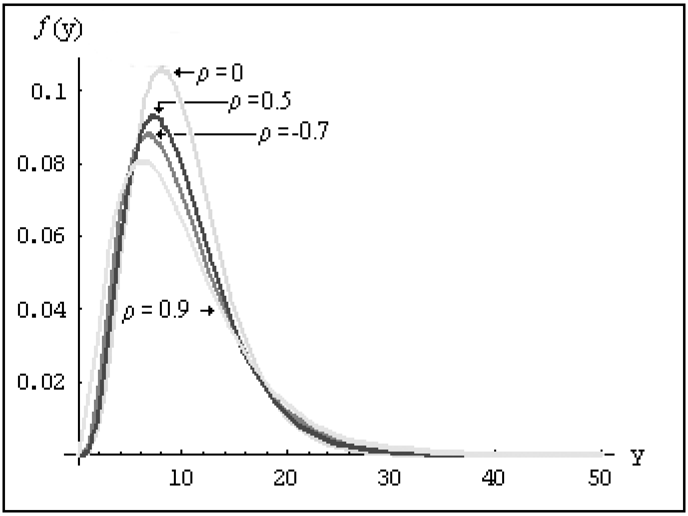

Fig. 1. Sum of Chi-square variables for $m = 5$ and various ρ values

chi-square random variable with $2m$ degrees of freedom. If $\rho = 0$, then the density function in Eq. 6 simplifies to that of a chi-square random variable with $2m$ degrees of freedom. This matches with the pdf of the sum of independent chi-square random variables (Johnson, Kotz, and Balakrishnan, 1994).

Theorem 3.2 Let Z have the density function given by Eq. 6. Then the Cumulative Density Function of Z is given by

$$F_Z(z) = \frac{(1-\rho^2)^{m/2}}{\Gamma(m)} \sum_{k=0}^{\infty} \frac{\Gamma((m+1)/2)}{\Gamma(k+(m+1)/2)(k!)} \left(\frac{\rho^2}{4}\right)^k \gamma\left(2k+m, \frac{z}{2-2\rho^2}\right), \quad (8)$$

$m > 2, -1 < \rho < 1$, and $\gamma(\alpha, x)$ is defined in Eq. 4.

Proof. It follows from Eq. 6 that the CDF of Z is given by

$$F_Z(z) = \frac{(1-\rho^2)^{-m/2}}{2^m\,\Gamma(m)} \sum_{k=0}^{\infty} \frac{\Gamma((m+1)/2)}{\Gamma(k+(m+1)/2)(k!)} \left(\frac{\rho^2}{(4-4\rho^2)^2}\right)^k I(k;m,\rho^2), \quad (9)$$

where $I(k;m,\rho^2) = \int_0^z t^{2k+m-1} \exp\left(\frac{-t}{2-2\rho^2}\right) dt.$

The above integral can be expressed as

$$I(k;m,\rho^2) = (2-2\rho^2)^{2k+m} \gamma\left(2k+m, \frac{z}{2-2\rho^2}\right), \quad (10)$$

where $\gamma(\alpha, x)$ is the incomplete gamma function defined in Eq. 4. By using Eq. 10 in Eq. 9, and a bit of simplification by Eq. 1, we have Eq. 8.

Percentage points of the distribution of Z can be calculated by Eq. 8 for particular values of the correlation coefficient ρ. By putting $\rho = 0$ in Theorem 3.1, we have the following corollary.

Corollary 3.1 Let U and V be two independent chi-square random variables. Then the Cumulative Density Function of $Z = U + V$ is given by

$$F_Z(z) = \frac{\gamma(m, z/2)}{\Gamma(m)}, \ m > 2,$$

where $\gamma(\alpha, x)$ is defined in Eq. 4.

THE CHARACTERISTIC FUNCTION OF THE SUM

Theorem 4.1 Let Z have the density function given by Eq. 6. Then for $-h < t < h$ where h is a positive number, the characteristic function of Z is given by

$$\phi_Z(t) = \left((1-2it)^2 + 4t^2\rho^2\right)^{-m/2}, \ m > 2 \textit{ and } -1 < \rho < 1. \quad (11)$$

Proof. By using Eq. 3 in Eq. 6, it follows that the characteristic function of the sum, Z, is given by the following:

$$\phi_Z(t) = \frac{\Gamma((m+1)/2)}{2^m\,\Gamma(m)(1-\rho^2)^{m/2}} \sum_{k=0}^{\infty} \frac{I(k;m,\rho^2)}{\Gamma(k+((m+1)/2))} \frac{\rho^{2k}}{(4-4\rho^2)^{2k} k!}, \quad (12)$$

where $I(k;m,\rho^2) = \int_0^{\infty} z^{m+2k-1} \exp\left[\left(it - (2-2\rho^2)^{-1}\right) z\right] dz.$

Evaluating the above gamma integral and a bit of simplification by Eq. 1, we have

$$\phi_Z(t) = \frac{(1-\rho^2)^{m/2}}{(1-2it+2it\rho^2)^m \Gamma(m/2)} \sum_{k=0}^{\infty} \frac{\Gamma(k+(m/2))}{(1-2it+2it\rho^2)^{2k}} \frac{\rho^{2k}}{k!}.$$

The above can be represented by

$$\phi_Z(t) = (1-\rho^2)^{m/2}\Big((1-2it+2it\rho^2)^2\Big)^{-m} {}_1F_0\Big(m/2; (1-2it+2it\rho^2)^{-2}\rho^2\Big),$$

which simplifies to $\phi_Z(t) = (1-\rho^2)^{m/2}\Big((1-2it+2it\rho^2)^2-\rho^2\Big)^{-m/2}$.

Since $\big(1-2it+2it\rho^2\big)^2-\rho^2 = (1-\rho^2)\Big((1-2it)^2+4t^2\rho^2\Big)$, it follows that $\phi_Z(t)$ is given by Eq. 11.

Note that in case $\rho = 0$, the characteristic function in Eq. 11 reduces to $\phi_Z(t) = (1-2it)^{-m}$ which is known to be the characteristic function of χ^2_{2m}, a chi-square random variable with $2m$ degrees of freedom.

The moment generating function of Z is given by $M_Z(t) = \phi_Z(t/i)$, which simplifies to $M_Z(t) = \Big((1-2t)^2-4t^2\rho^2\Big)^{-m}$ by Eq. 11. Letting $\zeta(t) = (1-2t)^2-4t^2\rho^2$, it can be checked that $\zeta'(t) = 8(1-\rho^2)t-4$,

$$M'_Z(t) = 2m\big(1+(2\rho^2-2)t\big)M'_Z(t)/\zeta(t), \tag{13}$$

and

$$M''_Z(t) = \frac{2m}{\zeta^2(t)}\big(1+(2\rho^2-2)t\big)M'_Z(t)\zeta(t)+(2\rho^2-2)M_Z(t)\zeta(t)+\big(1+(2-2\rho^2)t\big)\zeta'(t). \tag{14}$$

Since $\zeta(0) = 1$, $M_Z(0) = 1$, $\zeta'(0) = -4$, it follows from Eq. 13 and Eq. 14 that $M'_Z(0) = 2m$, and $M''_Z(0) = 4m(m+1+\rho^2)$, which are $E(Z)$ and $E(Z^2)$ respectively.

MOMENTS, COEFFICIENT OF SKEWNESS AND KURTOSIS FOR THE SUM

In this section, we provide some moment characteristics of the sum of two correlated chi-square variables.

Theorem 5.1 Let Z have the density function given by Eq. 6. Then for any number a, the $a-$th moment of Z denoted by $\mu'_a = E(Z^a)$ is given by

$$\mu'_a = \frac{2^a(1-\rho^2)^{a+(m/2)}\Gamma(m+a)}{\Gamma(m)} {}_2F_1\left(\frac{m+a}{2}, \frac{m+a+1}{2}; \frac{m+1}{2}; \rho^2\right), \tag{15}$$

where $m > 2$, $a > -2$, and $-1 < \rho < 1$.

Proof. It follows from Eq. 6 that

$$E(Z^a) = \frac{(1-\rho^2)^{-m/2}}{2^m\,\Gamma(m)} \sum_{k=0}^{\infty} \frac{\Gamma((m+1)/2))}{\Gamma(k+(m+1)/2)k!} \left(\frac{\rho^2}{(4-4\rho^2)^2}\right)^k I(k;m,\rho,a), \quad (16)$$

where

$$I(k;m,\rho,a) = \int_0^{\infty} z^{2k+m+a-1} \exp\left(\frac{-z}{2-2\rho^2}\right) dz. \quad (17)$$

By evaluating the above gamma integral (Eq. 17) by Eq. 1, and plugging the results into Eq. 16, we have

$$E(Z^a) = \frac{4^a(1-\rho^2)^{(m/2)+a}}{\Gamma(m/2)} \sum_{k=0}^{\infty} \frac{\Gamma(k+(m+a)/2)\Gamma(k+(m+a+1)/2)}{\Gamma(k+(m+1)/2)k!} \rho^{2k},$$

which can be represented by

$$E(Z^a) = 4^a(1-\rho^2)^{(m/2)+a} \frac{\Gamma((m+a)/2)\Gamma((m+a+1)/2)}{\Gamma(m/2)\Gamma((m+1)/2)}$$
$$\times {}_2F_1\left(\frac{m+a}{2}, \frac{m+a+1}{2}; \frac{m+1}{2}; \rho^2\right).$$

By using Eq. 1 in the above expression, we then have Eq. 15.

If $\rho = 0$, then it can be checked that Eq. 15 simplifies, as expected, to the $a-$th moment of $Z \sim \chi^2_{2m}$.

Corollary 5.1 Let Z have the density function given by Eq. 6. Then the first four raw moments of Z are respectively given by

$$(i)\ E(Z) = 2m, \quad (18)$$

$$(ii)\ E(Z^2) = 4m(m+1+\rho^2), \quad (19)$$

$$(iii)\ E(Z^3) = 8m(m+2)(m+1+3\rho^2), \quad (20)$$

$$(iv)\ E(Z^4) = 16m(m+2)\big((m+1)(m+3)+3\rho^2(\rho^2+2m+6)\big), \quad (21)$$

where $m > 2$ and $-1 < \rho < 1$.

Proof. The moments follow directly from $E(Z^a) = \int_0^\infty z^a f(z)dz$ where $f(z)$ is given by Eq. 6.

Theorem 5.2 Let Z have the density function given by Eq. 6. Then for any integer a, the $a-$th centered moment of Z, denoted by $\mu_a = E(Z - 2m)^a$, is given by

$$\mu_a = (-2m)^a \sum_{k=0}^{\infty} \frac{(-1)^k a^{\{k\}}}{k!} (2m)^k \mu'_k, \tag{22}$$

where $m > 2$, $-1 < \rho < 1$, and $a^{\{k\}}$ is defined by Eq. 2.

Proof. For any integer a, expanding $(Z - 2m)^a$ by binomial theorem, we have

$$(Z - 2m)^a = \sum_{k=0}^{\infty} \frac{a^{\{k\}}}{k!} Z^k (-2m)^{a-k}.$$

Since $(-2m)^{a-k} = (-2m)^a (-1)^k (2m)^{-k}$, we have

$$\mu_a = (-2m)^a \sum_{k=0}^{\infty} \frac{(-1)^k a^{\{k\}}}{k!} (2m)^{-k} E(Z^k).$$

which is the same as Eq. 22.

Corollary 5.2 Let Z have the density function given by Eq. 6. The centered moments of the sum, Z, of order 2, 3 and 4 are respectively given by

$$(i)\ \mu_2 = 4m(1 + \rho^2), \tag{23}$$

$$(ii)\ \mu_3 = 16m(1 + 3\rho^2), \tag{24}$$

$$(iii)\ \mu_4 = 48m[m(1 + \rho^2)^2 + 2(1 + 6\rho^2 + \rho^4)], \tag{25}$$

where $m > 2$ and$-1 < \rho < 1$.

Proof. Expanding $(Z - E(Z))^a$ by binomial theorem, taking expected value of the resulting expansion and using moments of Corollary 5.1, we have the expressions from Eq. 23 to Eq. 25.

The coefficient of skewness and kurtosis of a random variable Z are given by the moment ratios

$$\alpha_i(Z) = \mu_i \mu_2^{-i/2}, \ i = 3, 4. \tag{26}$$

Corollary 5.3 Let Z have the density function given by Eq. 6. Then the coefficient of skewness and kurtosis of Z are respectively given by

$$\alpha_3(Z) = \frac{2(1+3\rho^2)}{(1+\rho^2)^{3/2}\sqrt{m}}, \qquad \text{and} \tag{27}$$

$$\alpha_4(Z) = 3 + \frac{6(1+6\rho^2+\rho^4)]}{m(1+\rho^2)^2}, \tag{28}$$

where $m > 2$ and$-1 < \rho < 1$.

Proof. By Eq. 26, the coefficient of skewness is given by $\alpha_3(Z) = \mu_3/\mu_2^{3/2}$, which, by virtue of Eq. 23 and Eq. 24 simplifies to Eq. 27. By Eq. 26, the coefficient of kurtosis is given by $\alpha_4(Z) = \mu_4/\mu_2^2$, which, by virtue of Eq. 23 and Eq. 25 simplifies to Eq. 28.

If $\rho = 0$, then it follows from Corollary 5.3 that $\alpha_3(Z) = 2/(\sqrt{m})$, and $\alpha_4(Z) = 3 + (6/m)$, which matches with the coefficient of skewness and kurtosis of $Z \sim \chi^2_{2m}$. Further if the degrees of freedom m increases indefinitely, the coefficient of skewness and kurtosis converges, as expected, to 0 and 3 of the univariate normal distribution.

ESTIMATION OF COMMON VARIANCE OF BIVARIATE NORMAL DISTRIBUTION

Suppose that we want to estimate the common variance σ^2 of a bivariate normal population as discussed in Section 1. Since we have information from two samples based on bivariate normal distribution, the two sample variances S_1^2 and S_2^2 are correlated. Thus an estimator should be based on both the variances. The density function of the sample variances follows from Theorem 2.1.

Theorem 6.1 Let S_1^2 and S_2^2 be variances for a sample drawn from a bivariate normal distribution as discussed in the introduction. Then the joint density function of S_1^2 and S_2^2 is given by the following function:

$$f_{s_1^2,s_2^2}(s_1^2,s_2^2) = \frac{(s_1 s_2)^{m-2}\sigma_1^2\sigma_2^2}{2^m m^2\Gamma^2(m/2)(1-\rho^2)^{m/2}}\exp\left(-\frac{m[(s_1^2/\sigma_1^2)+(s_2^2/\sigma_2^2)]}{2-2\rho^2}\right) \times {}_0F_1\left(\frac{m}{2};\frac{m\rho^2[(s_1^2/\sigma_1^2)+(s_2^2/\sigma_2^2)]}{(2-2\rho^2)^2}\right), \tag{29}$$

where $m > 2$, $-1 < \rho < 1$ and ${}_0F_1(;b;z)$ is defined in Eq. 3.

An unbiased estimator of the common variance is presented in the following theorem.

Theorem 6.2 Let S_1^2 and S_2^2 be two sample variances based on a random sample of size $N = m+1$ bivariate normal distribution with common variance σ^2. Then an unbiased estimator of σ^2 is given by $\overline{S^2} = (S_1^2 + S_2^2)/2$.

Proof. Since $U = mS_1^2/\sigma^2 \sim \chi_m^2$, $V = mS_2^2/\sigma^2 \sim \chi_m^2$, and $E(S_1^2) = E(S_2^2) = \sigma^2$, we have $E\left(\overline{S^2}\right) = [E(S_1^2) + E(S_2^2)]/2 = \sigma^2$.

The density function of the estimator $\overline{S^2}$ can be derived from Theorem 6.1 as in Theorem 3.1 by assuming $\sigma_1^2 = \sigma_2^2$. Alternatively, it follows from Theorem 3.1 and is recorded below.

Theorem 6.3 Let S_1^2 and S_2^2 be two sample variances based on a random sample of size $N = m+1$ bivariate normal distribution with common variance σ^2. Then the density function of $\overline{s^2} = (s_1^2 + s_1^2)/2$, is given by

$$f_{\overline{s^2}}(\overline{s^2}) = \frac{m^m\left(\overline{s^2}\right)^{m-1}}{\Gamma(m)\sigma^{2m}(1-\rho^2)^{m/2}}\exp\left(\frac{-m\overline{s^2}}{\sigma^2(1-\rho^2)}\right){}_0F_1\left(\frac{m+1}{2};\frac{m^2\rho^2\left(\overline{s^2}\right)^2}{(2-2\rho^2)^2\sigma^4}\right), \tag{30}$$

where $m > 2$, $-1 < \rho < 1$ and $0 \le \overline{s^2} < \infty$.

Proof. In Theorem 3.1, $z = u + v$, which can be written as $z = ms_1^2\sigma^2 + ms_2^2\sigma^2$ or $z = 2m\overline{s^2}\sigma^2$. Then the density function of $\overline{s^2}$ follows from Theorem 3.1.

Theorem 6.4 A $100(1-\alpha)\%$ confidence interval for σ^2 is given by

$$P\left(\frac{2m\overline{S^2}}{c_2} < \sigma^2 < \frac{2m\overline{S^2}}{c_1}\right) = 1-\alpha, \tag{31}$$

where $\overline{S^2} = (S_1^2 + S_1^2)/2$, $c_1 = z_{1-(\alpha/2)}$ and $c_2 = z_{\alpha/2}$ such that $P(Z > c_1) = 1-\alpha/2$ and $P(Z > c_2) = \alpha/2$ with Z having the density function given by Eq. 6.

Proof. It follows from Eq. 6 that $P(c_1 < Z < c_2) = 1-\alpha$. Since $Z = U + V$, $Z = \dfrac{mS_1^2}{\sigma^2} + \dfrac{mS_2^2}{\sigma^2}$, $Z = \dfrac{2m\overline{S^2}}{\sigma^2}$, we have

$$P\left(c_1 < \frac{2m\,\overline{S^2}}{\sigma^2} < c_2\right) = 1 - \alpha, \quad \text{which is equivalent to Eq. 31.}$$

Tables 1 to 8 provide some percentage points for use in establishing the confidence interval presented in Theorem 6.4.

Table 1. Sum of Correlated Chi-square Percentage points for α =0.95

	absolute ρ										
m	0	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.85	0.9	0.95
2	0.711	0.682	0.658	0.627	0.586	0.533	0.500	0.463	0.418	0.365	0.296
3	1.635	1.576	1.527	1.462	1.378	1.270	1.206	1.131	1.046	0.947	0.832
4	2.733	2.642	2.568	2.470	2.344	2.184	2.089	1.983	1.863	1.729	1.581
5	3.940	3.819	3.722	3.592	3.427	3.220	3.099	2.965	2.817	2.654	2.478
6	5.226	5.076	4.957	4.798	4.596	4.347	4.203	4.044	3.871	3.684	3.483
7	6.571	6.394	6.253	6.067	5.832	5.544	5.379	5.198	5.003	4.794	4.570
8	7.962	7.759	7.598	7.387	7.121	6.797	6.613	6.413	6.197	5.967	5.723
9	9.390	9.164	8.984	8.748	8.454	8.097	7.894	7.676	7.441	7.192	6.928
10	10.851	10.601	10.403	10.145	9.823	9.435	9.215	8.979	8.727	8.460	8.177
11	12.338	12.066	11.851	11.571	11.223	10.805	10.571	10.318	10.049	9.764	9.464
12	13.848	13.555	13.324	13.023	12.650	12.205	11.955	11.687	11.402	11.101	10.784
13	15.379	15.065	14.818	14.497	14.101	13.629	13.365	13.082	12.782	12.465	12.132
14	16.928	16.594	16.332	15.992	15.573	15.076	14.798	14.501	14.186	13.854	13.505
15	18.493	18.140	17.863	17.505	17.064	16.542	16.251	15.941	15.612	15.265	14.901
16	20.072	19.700	19.409	19.034	18.572	18.026	17.723	17.399	17.056	16.695	16.318
17	21.664	21.275	20.970	20.577	20.095	19.527	19.211	18.874	18.518	18.144	17.752
18	23.269	22.862	22.544	22.134	21.633	21.042	20.714	20.365	19.996	19.609	19.203
19	24.884	24.460	24.129	23.703	23.183	22.570	22.231	21.870	21.489	21.089	20.670
20	26.509	26.069	25.725	25.284	24.745	24.111	23.761	23.388	22.995	22.582	22.151
21	28.144	27.687	27.332	26.874	26.318	25.664	25.302	24.918	24.513	24.089	23.645
22	29.788	29.315	28.947	28.475	27.901	27.227	26.855	26.460	26.043	25.607	25.150
23	31.439	30.951	30.572	30.085	29.493	28.800	28.417	28.012	27.584	27.136	26.668
24	33.098	32.595	32.204	31.703	31.095	30.383	29.990	29.573	29.135	28.675	28.195
25	34.764	34.246	33.844	33.329	32.704	31.974	31.571	31.144	30.695	30.224	29.733
26	36.437	35.905	35.492	34.963	34.322	33.573	33.161	32.724	32.264	31.782	31.280
27	38.116	37.569	37.146	36.604	35.947	35.180	34.758	34.312	33.841	33.349	32.836
28	39.801	39.284	38.806	38.251	37.579	36.795	36.363	35.907	35.427	34.924	34.400
29	41.492	40.917	40.473	39.905	39.217	38.416	37.975	37.509	37.019	36.506	35.972
30	43.188	42.477	42.145	41.564	40.862	40.044	39.594	39.119	38.619	38.096	37.551

Table 2. Sum of Correlated Chi-square Percentage points for $\alpha = 0.05$

	absolute ρ										
m	0	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.85	0.9	0.95
2	9.488	9.742	9.944	10.203	10.507	10.846	11.025	11.209	11.397	11.589	11.785
3	12.592	12.919	13.172	13.488	13.855	14.262	14.477	14.698	14.924	15.155	15.390
4	15.507	15.890	16.181	16.542	16.959	17.420	17.664	17.915	18.172	18.435	18.703
5	18.307	18.735	19.058	19.457	19.916	20.424	20.693	20.969	21.253	21.544	21.840
6	21.026	21.494	21.844	22.275	22.773	23.322	23.613	23.912	24.220	24.535	24.856
7	23.685	24.187	24.563	25.023	25.554	26.142	26.453	26.773	27.103	27.440	27.784
8	26.296	26.830	27.228	27.716	28.278	28.900	29.230	29.570	29.919	30.277	30.642
9	28.869	29.433	29.851	30.364	30.956	31.610	31.957	32.315	32.683	33.060	33.445
10	31.410	32.001	32.439	32.976	33.594	34.279	34.642	35.017	35.403	35.798	36.202
11	33.924	34.540	34.997	35.556	36.200	36.914	37.293	37.684	38.086	38.499	38.920
12	36.415	37.055	37.529	38.109	38.778	39.519	39.913	40.319	40.737	41.166	41.605
13	38.885	39.548	40.038	40.639	41.331	42.099	42.506	42.927	43.361	43.806	44.260
14	41.337	42.021	42.528	43.147	43.862	44.655	45.076	45.512	45.960	46.420	46.890
15	43.773	44.478	45.000	45.638	46.375	47.192	47.626	48.075	48.537	49.011	49.496
16	46.194	46.919	47.455	48.112	48.869	49.710	50.157	50.619	51.094	51.583	52.083
17	48.602	49.347	49.897	50.571	51.348	52.212	52.671	53.145	53.634	54.136	54.650
18	50.999	51.761	52.325	53.016	53.813	54.699	55.169	55.656	56.158	56.673	57.200
19	53.384	54.165	54.742	55.449	56.265	57.172	57.654	58.153	58.667	59.195	59.735
20	55.759	56.557	57.147	57.870	58.705	59.632	60.125	60.636	61.162	61.703	62.256
21	58.124	58.940	59.542	60.280	61.133	62.081	62.585	63.107	63.645	64.198	64.763
22	60.481	61.313	61.928	62.681	63.551	64.519	65.034	65.567	66.116	66.680	67.258
23	62.830	63.678	64.305	65.073	65.960	66.947	67.472	68.016	68.576	69.152	69.742
24	65.171	66.036	66.673	67.456	68.360	69.365	69.901	70.455	71.027	71.614	72.215
25	67.505	68.384	69.034	69.830	70.751	71.775	72.320	72.885	73.467	74.066	74.679
26	69.832	70.733	71.388	72.198	73.134	74.176	74.731	75.306	75.899	76.508	77.132
27	72.153	73.063	73.734	74.558	75.510	76.570	77.135	77.719	78.322	78.942	79.577
28	74.468	75.369	76.075	76.911	77.879	78.956	79.530	80.125	80.738	81.368	82.014
29	76.778	77.716	78.409	79.258	80.241	81.336	81.919	82.523	83.146	83.786	84.443
30	79.082	79.427	80.737	81.600	82.597	83.708	84.300	84.914	85.547	86.197	86.864

Table 3. Sum of Correlated Chi-square Percentage points for $\alpha = 0.975$

	absolute ρ										
m	0	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.85	0.9	0.95
2	0.484	0.464	0.447	0.425	0.396	0.357	0.334	0.307	0.275	0.236	0.183
3	1.237	1.189	1.150	1.098	1.030	0.941	0.888	0.826	0.754	0.668	0.562
4	2.180	2.101	2.038	1.953	1.843	1.703	1.618	1.522	1.412	1.284	1.136
5	3.247	3.138	3.051	2.934	2.783	2.592	2.478	2.351	2.206	2.044	1.862
6	4.404	4.265	4.154	4.006	3.817	3.578	3.438	3.282	3.108	2.916	2.704
7	5.629	5.462	5.328	5.150	4.924	4.642	4.477	4.295	4.095	3.875	3.636
8	6.908	6.713	6.558	6.352	6.090	5.767	5.580	5.374	5.149	4.905	4.642
9	8.231	8.010	7.833	7.600	7.306	6.945	6.736	6.509	6.261	5.994	5.707
10	9.591	9.344	9.148	8.889	8.564	8.166	7.938	7.689	7.421	7.131	6.822
11	10.982	10.711	10.496	10.212	9.857	9.424	9.178	8.910	8.621	8.311	7.981
12	12.401	12.106	11.872	11.565	11.181	10.716	10.451	10.165	9.857	9.528	9.178
13	13.844	13.526	13.274	12.944	12.533	12.036	11.755	11.451	11.125	10.777	10.407
14	15.308	14.967	14.698	14.347	13.909	13.382	13.084	12.763	12.420	12.054	11.666
15	16.791	16.429	16.143	15.770	15.307	14.751	14.438	14.100	13.740	13.356	12.951
16	18.291	17.908	17.605	17.212	16.725	16.141	15.812	15.459	15.082	14.682	14.260
17	19.806	19.403	19.085	18.671	18.160	17.549	17.206	16.838	16.445	16.029	15.590
18	21.336	20.912	20.579	20.146	19.612	18.975	18.617	18.234	17.827	17.395	16.939
19	22.879	22.435	22.087	21.636	21.079	20.416	20.045	19.648	19.225	18.778	18.307
20	24.433	23.971	23.608	23.138	22.560	21.872	21.487	21.076	20.639	20.177	19.691
21	25.999	25.518	25.141	24.653	24.053	23.341	22.944	22.519	22.068	21.591	21.090
22	27.575	27.076	26.685	26.179	25.559	24.823	24.413	23.975	23.510	23.020	22.504
23	29.160	28.643	28.239	27.716	27.076	26.317	25.894	25.443	24.965	24.461	23.931
24	30.755	30.222	29.802	29.263	28.603	27.821	27.387	26.923	26.432	25.914	25.370
25	32.357	31.806	31.375	30.820	30.139	29.336	28.890	28.414	27.910	27.379	26.822
26	33.968	33.406	32.956	32.385	31.685	30.861	30.403	29.915	29.398	28.854	28.284
27	35.586	35.001	34.545	33.958	33.240	32.394	31.925	31.425	30.896	30.339	29.756
28	37.212	36.574	36.142	35.539	34.803	33.936	33.456	32.944	32.404	31.834	31.238
29	38.844	38.226	37.746	37.128	36.374	35.487	34.995	34.472	33.920	33.338	32.729
30	40.482	39.678	39.356	38.723	37.952	37.045	36.542	36.008	35.444	34.851	34.229

Table 4. Sum of Correlated Chi-square Percentage points for $\alpha = 0.025$

	absolute ρ										
m	0	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.85	0.9	0.95
2	11.143	11.570	11.892	12.283	12.725	13.202	13.451	13.704	13.962	14.223	14.488
3	14.449	14.963	15.342	15.799	16.314	16.871	17.162	17.459	17.762	18.069	18.381
4	17.535	18.112	18.535	19.044	19.618	20.239	20.564	20.897	21.236	21.581	21.932
5	20.483	21.114	21.573	22.125	22.749	23.426	23.780	24.143	24.514	24.892	25.276
6	23.337	24.013	24.504	25.095	25.762	26.488	26.869	27.260	27.659	28.066	28.479
7	26.119	26.835	27.355	27.980	28.688	29.459	29.864	30.279	30.704	31.137	31.578
8	28.845	29.598	30.144	30.801	31.546	32.358	32.785	33.223	33.672	34.130	34.596
9	31.526	32.313	32.883	33.570	34.349	35.200	35.647	36.107	36.578	37.059	37.548
10	34.170	34.987	35.580	36.295	37.106	37.993	38.460	38.940	39.432	39.934	40.446
11	36.781	37.628	38.242	38.983	39.825	40.746	41.232	41.731	42.243	42.766	43.298
12	39.364	40.239	40.874	41.640	42.511	43.465	43.968	44.485	45.016	45.558	46.111
13	41.923	42.825	43.479	44.268	45.167	46.153	46.672	47.207	47.756	48.317	48.889
14	44.461	45.388	46.060	46.873	47.798	48.814	49.349	49.901	50.467	51.046	51.637
15	46.979	47.930	48.621	49.455	50.406	51.451	52.002	52.570	53.153	53.749	54.358
16	49.480	50.455	51.163	52.018	52.994	54.066	54.633	55.216	55.815	56.428	57.054
17	51.966	52.963	53.687	54.563	55.563	56.663	57.243	57.842	58.457	59.086	59.728
18	54.437	55.456	56.196	57.092	58.115	59.241	59.836	60.449	61.079	61.724	62.382
19	56.896	57.935	58.691	59.607	60.652	61.803	62.412	63.039	63.684	64.344	65.018
20	59.342	60.402	61.173	62.107	63.175	64.350	64.972	65.614	66.273	66.948	67.637
21	61.777	62.857	63.643	64.595	65.684	66.884	67.519	68.174	68.847	69.536	70.240
22	64.202	65.301	66.101	67.072	68.181	69.405	70.052	70.720	71.407	72.110	72.829
23	66.617	67.735	68.549	69.537	70.667	71.913	72.573	73.254	73.954	74.671	75.404
24	69.023	70.161	70.988	71.992	73.142	74.411	75.083	75.776	76.489	77.220	77.967
25	71.420	72.575	73.417	74.438	75.607	76.898	77.582	78.287	79.013	79.757	80.518
26	73.810	74.997	75.837	76.875	78.063	79.375	80.071	80.788	81.527	82.284	83.057
27	76.192	77.382	78.250	79.303	80.510	81.844	82.550	83.280	84.030	84.800	85.587
28	78.567	79.901	80.654	81.723	82.949	84.303	85.021	85.762	86.525	87.307	88.106
29	80.936	82.159	83.052	84.136	85.380	86.754	87.483	88.235	89.010	89.804	90.616
30	83.298	84.151	85.443	86.542	87.803	89.197	89.937	90.701	91.487	92.293	93.117

Table 5. Sum of Correlated Chi-square Percentage points for $\alpha = 0.99$

	absolute ρ										
m	0	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.85	0.9	0.95
2	0.297	0.284	0.274	0.259	0.241	0.216	0.202	0.184	0.164	0.138	0.104
3	0.872	0.836	0.807	0.768	0.717	0.651	0.611	0.564	0.509	0.441	0.354
4	1.647	1.583	1.532	1.463	1.373	1.258	1.188	1.107	1.013	0.901	0.764
5	2.558	2.466	2.391	2.291	2.161	1.994	1.894	1.780	1.649	1.496	1.317
6	3.571	3.448	3.350	3.218	3.048	2.832	2.703	2.557	2.391	2.202	1.987
7	4.660	4.509	4.387	4.224	4.015	3.751	3.594	3.419	3.222	3.000	2.753
8	5.812	5.632	5.487	5.294	5.047	4.738	4.555	4.352	4.126	3.875	3.597
9	7.015	6.806	6.639	6.417	6.134	5.781	5.574	5.345	5.092	4.813	4.508
10	8.260	8.025	7.836	7.586	7.267	6.873	6.643	6.390	6.111	5.807	5.475
11	9.542	9.280	9.070	8.793	8.441	8.007	7.755	7.479	7.177	6.847	6.491
12	10.856	10.568	10.338	10.034	9.650	9.177	8.905	8.607	8.282	7.929	7.549
13	12.198	11.884	11.634	11.305	10.890	10.381	10.088	9.769	9.422	9.048	8.645
14	13.565	13.226	12.957	12.603	12.157	11.613	11.302	10.962	10.595	10.199	9.774
15	14.954	14.591	14.303	13.924	13.450	12.872	12.542	12.183	11.796	11.379	10.933
16	16.362	15.976	15.670	15.268	14.765	14.154	13.806	13.429	13.022	12.586	12.120
17	17.789	17.380	17.056	16.631	16.101	15.458	15.093	14.698	14.272	13.817	13.331
18	19.233	18.801	18.459	18.013	17.456	16.782	16.400	15.988	15.544	15.069	14.565
19	20.691	20.238	19.879	19.411	18.828	18.124	17.726	17.296	16.835	16.343	15.819
20	22.164	21.689	21.314	20.824	20.216	19.483	19.069	18.623	18.145	17.634	17.093
21	23.650	23.154	22.762	22.252	21.619	20.858	20.428	19.966	19.471	18.944	18.384
22	25.148	24.631	24.224	23.693	23.036	22.247	21.802	21.324	20.813	20.269	19.693
23	26.657	26.120	25.697	25.147	24.466	23.650	23.190	22.697	22.170	21.609	21.016
24	28.177	27.623	27.182	26.612	25.907	25.065	24.591	24.083	23.540	22.964	22.354
25	29.707	29.130	28.677	28.088	27.361	26.492	26.005	25.482	24.924	24.332	23.706
26	31.246	30.657	30.182	29.574	28.825	27.931	27.430	26.892	26.319	25.712	25.070
27	32.793	32.179	31.696	31.070	30.299	29.380	28.865	28.314	27.727	27.104	26.447
28	34.350	33.818	33.219	32.576	31.783	30.839	30.311	29.746	29.144	28.507	27.835
29	35.914	35.262	34.751	34.089	33.275	32.308	31.767	31.188	30.573	29.921	29.234
30	37.485	36.959	36.290	35.611	34.777	33.786	33.232	32.640	32.010	31.344	30.643

Table 6. Sum of Correlated Chi-square Percentage points for $\alpha = 0.01$

	absolute ρ										
m	0	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.85	0.9	0.95
2	13.277	13.970	14.460	15.032	15.657	16.318	16.658	17.003	17.352	17.705	18.062
3	16.812	17.602	18.156	18.803	19.513	20.267	20.657	21.053	21.456	21.863	22.274
4	20.090	20.951	21.555	22.261	23.040	23.870	24.300	24.738	25.184	25.635	26.092
5	23.209	24.129	24.774	25.530	26.367	27.263	27.728	28.203	28.685	29.175	29.671
6	26.217	27.187	27.869	28.670	29.558	30.512	31.009	31.515	32.031	32.555	33.086
7	29.141	30.157	30.871	31.712	32.648	33.655	34.179	34.715	35.262	35.817	36.380
8	32.000	33.056	33.800	34.679	35.658	36.714	37.265	37.828	38.402	38.987	39.580
9	34.805	35.899	36.672	37.585	38.605	39.706	40.281	40.870	41.470	42.082	42.703
10	37.566	38.696	39.494	40.440	41.498	42.642	43.240	43.853	44.478	45.115	45.762
11	40.289	41.452	42.275	43.252	44.346	45.531	46.151	46.786	47.435	48.096	48.768
12	42.980	44.174	45.021	46.027	47.155	48.379	49.020	49.677	50.348	51.032	51.728
13	45.642	46.866	47.735	48.769	49.931	51.191	51.852	52.530	53.222	53.929	54.647
14	48.278	49.531	50.422	51.483	52.676	53.972	54.652	55.350	56.063	56.790	57.531
15	50.892	52.173	53.085	54.171	55.395	56.725	57.423	58.140	58.873	59.621	60.383
16	53.486	54.793	55.725	56.836	58.089	59.453	60.169	60.904	61.656	62.424	63.206
17	56.061	57.394	58.345	59.481	60.762	62.158	62.891	63.644	64.415	65.202	66.003
18	58.619	59.978	60.947	62.106	63.415	64.841	65.591	66.362	67.151	67.956	68.777
19	61.162	62.545	63.533	64.714	66.049	67.506	68.272	69.060	69.866	70.690	71.529
20	63.691	65.097	66.103	67.306	68.667	70.154	70.935	71.739	72.563	73.404	74.261
21	66.206	67.635	68.658	69.883	71.270	72.785	73.582	74.402	75.242	76.100	76.975
22	68.710	70.161	71.201	72.447	73.858	75.401	76.213	77.048	77.904	78.779	79.671
23	71.201	72.675	73.731	74.998	76.433	78.003	78.829	79.680	80.552	81.443	82.352
24	73.683	75.182	76.250	77.536	78.995	80.592	81.433	82.298	83.186	84.093	85.018
25	76.154	77.669	78.758	80.064	81.546	83.168	84.023	84.903	85.806	86.729	87.670
26	78.616	80.158	81.255	82.581	84.085	85.733	86.602	87.496	88.414	89.352	90.309
27	81.069	82.625	83.744	85.088	86.614	88.287	89.170	90.078	91.010	91.963	92.936
28	83.513	85.184	86.223	87.585	89.134	90.831	91.727	92.649	93.595	94.563	95.551
29	85.950	87.545	88.693	90.074	91.644	93.365	94.274	95.209	96.170	97.152	98.155
30	88.379	88.144	91.156	92.554	94.145	95.890	96.811	97.760	98.734	99.731	100.748

Table 7. Sum of Correlated Chi-square Percentage points for $\alpha = 0.995$

	absolute ρ										
m	0	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.85	0.9	0.95
2	0.207	0.198	0.190	0.180	0.167	0.150	0.139	0.127	0.113	0.094	0.070
3	0.676	0.647	0.624	0.593	0.552	0.499	0.467	0.430	0.385	0.330	0.258
4	1.344	1.291	1.247	1.189	1.113	1.014	0.954	0.885	0.804	0.705	0.580
5	2.156	2.075	2.009	1.920	1.805	1.657	1.568	1.465	1.346	1.204	1.033
6	3.074	2.964	2.874	2.755	2.600	2.402	2.283	2.148	1.992	1.811	1.599
7	4.075	3.935	3.823	3.672	3.478	3.231	3.083	2.916	2.727	2.510	2.261
8	5.142	4.974	4.838	4.657	4.424	4.129	3.954	3.757	3.536	3.286	3.004
9	6.265	6.068	5.909	5.698	5.427	5.086	4.885	4.660	4.409	4.129	3.816
10	7.434	7.208	7.028	6.787	6.479	6.094	5.869	5.617	5.338	5.029	4.687
11	8.643	8.390	8.187	7.918	7.575	7.147	6.897	6.621	6.315	5.979	5.609
12	9.886	9.606	9.382	9.085	8.707	8.239	7.966	7.666	7.335	6.973	6.577
13	11.160	10.854	10.609	10.285	9.874	9.366	9.071	8.748	8.393	8.006	7.585
14	12.461	12.129	11.863	11.513	11.070	10.524	10.208	9.862	9.484	9.073	8.629
15	13.787	13.429	13.143	12.767	12.293	11.710	11.374	11.007	10.607	10.173	9.705
16	15.134	14.751	14.446	14.045	13.540	12.921	12.566	12.178	11.757	11.301	10.810
17	16.501	16.094	15.770	15.345	14.810	14.156	13.782	13.374	12.932	12.455	11.942
18	17.887	17.456	17.113	16.664	16.100	15.412	15.019	14.592	14.130	13.632	13.099
19	19.289	18.834	18.474	18.001	17.408	16.688	16.277	15.832	15.350	14.832	14.278
20	20.707	20.229	19.851	19.355	18.735	17.983	17.554	17.090	16.590	16.052	15.478
21	22.139	21.638	21.243	20.724	20.077	19.294	18.849	18.367	17.848	17.291	16.698
22	23.584	23.061	22.648	22.108	21.435	20.621	20.159	19.660	19.123	18.548	17.935
23	25.041	24.497	24.067	23.506	22.807	21.963	21.485	20.968	20.414	19.821	19.189
24	26.511	25.950	25.499	24.916	24.191	23.318	22.824	22.292	21.720	21.109	20.460
25	27.991	27.404	26.942	26.338	25.589	24.687	24.178	23.629	23.040	22.412	21.745
26	29.481	28.901	28.395	27.772	26.998	26.068	25.543	24.978	24.373	23.728	23.044
27	30.981	30.354	29.859	29.216	28.418	27.461	26.921	26.341	25.719	25.057	24.356
28	32.491	32.014	31.333	30.670	29.848	28.864	28.310	27.714	27.077	26.399	25.680
29	34.008	33.341	32.816	32.133	31.289	30.278	29.709	29.099	28.446	27.752	27.017
30	35.535	31.705	34.308	33.606	32.739	31.702	31.119	30.494	29.826	29.116	28.365

Table 8. Sum of Correlated Chi-square Percentage points for $\alpha = 0.005$

	absolute ρ										
m	0	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.85	0.9	0.95
2	14.860	15.778	16.402	17.111	17.875	18.674	19.084	19.498	19.917	20.339	20.765
3	18.548	19.565	20.257	21.047	21.904	22.807	23.271	23.742	24.218	24.700	25.186
4	21.955	23.047	23.790	24.646	25.578	26.564	27.073	27.590	28.114	28.644	29.180
5	25.188	26.341	27.129	28.039	29.036	30.094	30.641	31.198	31.763	32.335	32.914
6	28.300	29.505	30.332	31.291	32.345	33.468	34.049	34.641	35.243	35.853	36.471
7	31.319	32.572	33.434	34.438	35.544	36.725	37.337	37.962	38.597	39.242	39.895
8	34.267	35.562	36.458	37.502	38.656	39.891	40.532	41.187	41.853	42.530	43.216
9	37.157	38.492	39.417	40.499	41.698	42.984	43.652	44.335	45.030	45.737	46.453
10	39.997	41.370	42.324	43.441	44.682	46.015	46.709	47.418	48.141	48.876	49.621
11	42.796	44.204	45.185	46.336	47.617	48.995	49.713	50.447	51.196	51.958	52.731
12	45.559	47.000	48.007	49.191	50.509	51.930	52.671	53.429	54.203	54.990	55.789
13	48.290	49.764	50.795	52.010	53.364	54.826	55.589	56.370	57.168	57.979	58.803
14	50.993	52.498	53.553	54.797	56.187	57.688	58.472	59.275	60.095	60.930	61.778
15	53.672	55.206	56.284	57.556	58.979	60.518	61.323	62.147	62.989	63.847	64.718
16	56.328	57.891	58.990	60.290	61.746	63.322	64.146	64.991	65.854	66.733	67.627
17	58.964	60.555	61.675	63.002	64.489	66.100	66.943	67.807	68.691	69.592	70.507
18	61.581	63.199	64.340	65.692	67.209	68.855	69.716	70.600	71.504	72.425	73.362
19	64.181	65.825	66.986	68.363	69.910	71.589	72.468	73.371	74.294	75.235	76.192
20	66.766	68.435	69.615	71.017	72.593	74.304	75.201	76.121	77.063	78.024	79.001
21	69.336	71.030	72.229	73.655	75.258	77.001	77.915	78.853	79.813	80.793	81.789
22	71.893	73.611	74.828	76.277	77.907	79.681	80.612	81.567	82.545	83.543	84.559
23	74.437	76.178	77.414	78.885	80.542	82.346	83.293	84.265	85.261	86.277	87.311
24	76.969	78.742	79.987	81.480	83.163	84.997	85.959	86.948	87.961	88.994	90.047
25	79.490	81.277	82.548	84.063	85.772	87.634	88.612	89.617	90.646	91.697	92.768
26	82.001	83.876	85.098	86.634	88.368	90.258	91.251	92.272	93.318	94.386	95.474
27	84.502	86.333	87.637	89.194	90.952	92.870	93.878	94.915	95.977	97.061	98.166
28	86.994	88.836	90.167	91.744	93.526	95.471	96.494	97.546	98.623	99.724	100.846
29	89.477	91.351	92.687	94.284	96.090	98.061	99.099	100.165	101.258	102.375	103.514
30	91.952	93.732	95.198	96.815	98.644	100.642	101.693	102.774	103.882	105.015	106.170

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التوزيع المضبوط لمجموع متغيرين مترابطين من نوع كاي – تربيع وتطبيقاته

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نستطيع معرفة التوزيع المضبوط لمجموع متغيرين من نوع كاي – تربيع إذا كانت المتغيرات مستقلة. نقوم في هذا البحث بإيجاد التوزيع المضبوط لمجموع متغيرين مترابطين من نوع كاي – تربيع وذلك عندما تكون المتغيرات مترابطة من خلال توزيع كاي – تربيع ثنائي التغير. ونحصل على بعض خصائص التوزيع مثل: الدالة المميزة، دالة التوزيع التراكمي، العزم الخام، العزم الوسطي الارتكاز، معامل التخالف والتفلطح. كما نقوم أيضاً بعرض الرسم البياني لدالة الكثافة.

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