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On logarithmic averages of sequences and its applications

Ümit Totur[1,*], Muhammet A. Okur[2]

[1]*Dept. of Mathematics, Adnan Menderes University, 09010, Aydın, Turkey*

[2]*Dept. of Mathematics, Adnan Menderes University, 09010, Aydın, Turkey*
E-mail: mali.okur@adu.edu.tr, mali.okur2@gmail.com

[*]*Corresponding author: utotur@adu.edu.tr, utotur@yahoo.com*

Abstract

In this paper, we investigate summability methods of logarithmic averages of the numerical sequences and its applications such as Tauberian type theorems.

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1. Introduction

The main objective of Tauberian theory is to obtain convergence of such sequences out of the existence of certain limits by some summability methods and some additional conditions, which are called Tauberian condition. Furthermore, a Tauberian theorem is a theorem, which deduces convergence of sequences follow from a summability method and some Tauberian conditions. One of the summability methods, which is used in theorems is the logarithmic summability method. Recently, there have been many studies about some summability methods and Tauberian theorem (Çanak, 2015; Çanak & Totur, 2015; Et *et al.* 2014a; Et *et al.* 2014b; Totur & Okur, 2015)).

The logarithmic summability method was first introduced by Hardy (1949). Naturally, Tauberian type theorems of this method were established by various authors (Kwee, 1966; Móricz, 2013). The logarithmic average or logarithmic mean of a sequence is more general than its arithmetic average. So, Tauberian theorems in this paper are more general than their arithmetic analogues.

Here, we introduce the logarithmic summability methods.

Given a sequence (u_n) of real numbers, the logarithmic mean of order 1 of (u_n) is defined (Hardy, 1949) by

$$\ell_{n,1}(u) = \frac{1}{\gamma_{n,1}}\sum_{k=0}^{n}\frac{u_k}{k+1},$$

where

$$\gamma_{n,1} = \sum_{k=0}^{n}\frac{1}{k+1} \sim \log n.$$

A sequence (u_n) is said to be logarithmic summable of order 1, in short, $(\ell,1)$ converges, to a finite number s if

$$\lim_{n\to\infty} \ell_{n,1}(u) = s. \tag{1}$$

Now, we introduce the $(\ell,2)$ summability method. The logarithmic mean of order 2 of (u_n) is defined (Móricz, 2006) by

$$\ell_{n,2}(u) = \frac{1}{\gamma_{n,2}}\sum_{k=0}^{n}\frac{u_k}{(k+1)\gamma_{k,1}},$$

where

$$\gamma_{n,2} = \sum_{k=0}^{n}\frac{1}{(k+1)\gamma_{k,1}} \sim \log(\log n).$$

A sequence (u_n) is logarithmic summable of order 2, in short; $(\ell,2)$ converges, to a finite number s if

$$\lim_{n\to\infty} \ell_{n,2}(u) = s. \tag{2}$$

The convergence of (u_n) to s implies that the limits (1) and (2) also exist. In other words, the logarithmic methods of order 1 and 2 are regular methods. Additionally, every $(\ell,1)$ summable sequence is $(\ell,2)$ summable to same value (Móricz, 2006). These implications can be summarized as in the following diagram:

$$\text{convergence} \Rightarrow (\ell,1) \text{ convergence} \Rightarrow (\ell,2) \text{ convergence}.$$

However, the converse implications are not always true. Namely, the $(\ell,1)$ convergence of (u_n) doesn't imply convergence of (u_n), and the $(\ell,2)$ convergence of (u_n) doesn't imply $(\ell,1)$ convergence of (u_n).

Example 1.Consider the sequence $(u_n)=(2(-1)^n n+(-1)^n)$. The sequence is $(\ell,1)$ convergent but not ordinary convergent. If we get the arithmetic mean of (u_n), we obtain that $\sigma_{n,1}(u)=\frac{1}{n+1}\sum_{k=0}^{n}u_k \sim (-1)^n$. Then, the logarithmic mean of order 1 of a sequence can be written in terms of $(\sigma_{n,1}(u))$ as follows:

$$\ell_{n,1}(u)=\frac{\sigma_{n,1}(u)}{\gamma_{n,1}}+\frac{1}{\gamma_{n,1}}\sum_{k=1}^{n}\frac{\sigma_{k-1,1}(u)}{k}\cdot \tag{3}$$

Hence (u_n) is $(\ell,1)$ convergent to 0.

Example 2. Consider the sequence $(u_n)=((-1)^n((n+1)\log(n+1)+n\log n))$. The sequence is $(\ell,2)$ convergent but not $(\ell,1)$ convergent. If we get the arithmetic mean of (u_n), we obtain that $\sigma_{n,1}(u)\sim(-1)^n\log n$. If we write $\sigma_n(u)$ in (3), then we have $\ell_{n,1}(u)\sim(-1)^n$, and we obtain that (u_n) is not $(\ell,1)$ convergent. Also, the logarithmic mean of order 2 of a sequence can be written in terms of $(\ell_{n,1}(u))$ as follows:

$$\ell_{n,2}(u)=\frac{\ell_{n,1}(u)}{\gamma_{n,2}}+\frac{1}{\gamma_{n,2}}\sum_{k=1}^{n}\frac{\ell_{k-1,1}(u)}{k\log(k+1)}\cdot \tag{4}$$

Therefore, the sequence (u_n) is $(\ell,2)$ convergent to 0.

We are interested in the "converse" case of these assertions. The converse cases are true under suitable Tauberian conditions. The first classical Tauberian theorem for the logarithmic summability method of order 1 obtained by Ishiguro (1963).

Theorem 3. If (u_n) is $(\ell,1)$ convergent to s and $n\log n\Delta u_n=o(1)$, where $\Delta u_n=u_n-u_{n-1}$, then (u_n) converges to the same value.

Later, Kwee (1966) proved the following theorem that was generalized to Theorem 3.

Theorem 4. If (u_n) is $(\ell,1)$ convergent to s and

$$\liminf(u_m-u_n)\geq 0 \text{ when } m>n\to\infty \text{ and } \frac{\log m}{\log n}\to 1,$$

then (u_n) converges to the same value.

Notice that Tauberian condition in Theorem 4 is the slow decrease of (u_n) with respect to $(\ell,1)$.

As the $(C,1)$ summability method, the classical one-sided Tauberian condition $n\log n\Delta u_n\geq -C$, for some $C>0$ implies that the slowly decrease of (u_n) with respect to $(\ell,1)$. Indeed,

$$u_m-u_n=\sum_{k=n+1}^{m}\Delta u_k\geq\sum_{k=n+1}^{m}\frac{-C}{k\log k}\geq -C\log\left(\frac{\log m}{\log n}\right).$$

Taking liminf of both-sides when $m>n\to\infty$ and $\frac{\log m}{\log n}\to 1$, it is obtained that (u_n) is slowly decreasing with respect to $(\ell,1)$.

Móricz (2013) presented that a sequence (u_n) is slowly decreasing with respect to $(\ell,1)$ if and only if the condition

$$\limsup_{\lambda\to 1^+}\liminf_{n\to\infty}\min_{n<j\leq n^\lambda}(u_j-u_n)\geq 0.$$

Note that the conditions $n<j\leq n^\lambda$ and $\log n<\log j\leq \lambda\log n$ are equivalent.

Móricz (2013) improved Kwee's Tauberian condition as follows:

$$\limsup_{\lambda\to 1^+}\liminf_{n\to\infty}\frac{1}{([n^\lambda]-n)\gamma_{n,1}}\sum_{j=n+1}^{[n^\lambda]}\frac{u_j-u_n}{j}\geq 0,$$

$$\limsup_{\lambda\to 1^-}\liminf_{n\to\infty}\frac{1}{(n-[n^\lambda])\gamma_{n,1}}\sum_{j=[n^\lambda]+1}^{n}\frac{u_n-u_j}{j}\geq 0,$$

where $[\cdot]$ denotes integer part.

A slowly decreasing sequence (u_n) with respect to $(\ell,2)$ is defined (Móricz, 2004) by

$$\lim_{\lambda\to 1^+}\liminf_{n\to\infty}\min_{n<j\leq e^{(\log n)^\lambda}}(u_j-u_n)\geq 0. \tag{5}$$

It can be easily seen that the conditions $n<j\leq e^{(\log n)^\lambda}$ and $\log(\log n)<\log(\log j)\leq\lambda\log(\log n)$ are equivalent.

The condition (5) can be equivalently reformulated as follows:

$$\lim_{\lambda\to 1^-}\liminf_{n\to\infty}\min_{e^{(\log n)^\lambda}\leq j<n}(u_j-u_n)\geq 0. \tag{6}$$

The de la Vallée Poussin means of (u_n) with respect to $(\ell,2)$ are defined by

$$\tau^{>}_{[e^{(\log n)^\lambda}],2}(u)=\frac{1}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^\lambda}]}\frac{u_k}{(k+1)\gamma_{k,1}},$$

for $\lambda>1$, and

$$\tau^{<}_{[e^{(\log n)^\lambda}],2}(u)=\frac{1}{\gamma_{n,2}-\gamma_{[e^{(\log n)^\lambda}],2}}\sum_{k=[e^{(\log n)^\lambda}]+1}^{n}\frac{u_k}{(k+1)\gamma_{k,1}},$$

for $0<\lambda<1$, where $[\cdot]$ denotes integer part.

The main goal of this paper is to establish some Tauberian theorems for the logarithmic summability method order 2 with a different viewpoint and to generalize above-mentioned classical Tauberian theorems and Móricz's theorem for the logarithmic summability method order 1 and 2.

2. Main results for $(\ell,2)$ summability method

In this section, we give Tauberian theorems for the $(\ell,2)$ summability method. First of all, we represent the most prominent Tauberian theorem.

Theorem 5. If (u_n) is $(\ell,2)$ convergent to s and

$$V_{n,2}(\Delta u)=o(1)\text{, where } V_{n,2}(\Delta u)=\frac{1}{\gamma_{n,2}}\sum_{k=1}^{n}\gamma_{k-1,2}\Delta u_k,$$

then (u_n) converges to the same value.

Proof. It is sufficient to show the following identity in order to prove the theorem. The difference of a sequence and its logarithmic mean order 2 is represented by the identity

$$u_n-\ell_{n,2}(u)=V_{n,2}(\Delta u). \tag{7}$$

Indeed,

$$\sum_{k=0}^{n}\frac{u_k}{(k+1)\gamma_{k,1}}=\sum_{k=0}^{n}(\gamma_{k,2}-\gamma_{k-1,2})u_k$$

$$=\sum_{k=0}^{n}\gamma_{k,2}u_k-\sum_{k=0}^{n}\gamma_{k-1,2}u_k$$

$$=\sum_{k=1}^{n+1}\gamma_{k-1,2}u_{k-1}-\sum_{k=0}^{n}\gamma_{k-1,2}u_k$$

$$=\sum_{k=1}^{n}\gamma_{k-1,2}(u_{k-1}-u_k)+\gamma_{n,2}u_n.$$

where $\gamma_{-1,2}=0$. After multiplying both sides of equality by $\frac{1}{\gamma_{n,2}}$, we obtain

$$\frac{1}{\gamma_{n,2}}\sum_{k=0}^{n}\frac{u_k}{(k+1)\gamma_{k,1}}=-\frac{1}{\gamma_{n,2}}\sum_{k=1}^{n}\gamma_{k-1,2}\Delta u_k+u_n,$$

and

$$u_n-\frac{1}{\gamma_{n,2}}\sum_{k=0}^{n}\frac{u_k}{(k+1)\gamma_{k,1}}=\frac{1}{\gamma_{n,2}}\sum_{k=1}^{n}\gamma_{k-1,2}\Delta u_k.$$

Therefore, the identity (7) is satisfied.

By the identity (7) and the convergence of the sequence $(\ell_{n,2}(u))$, the proof is completed.

Next corollary generalizes Kwee's Tauberian theorem to the $(\ell,2)$ summability method.

Corollary 6. If (u_n) is $(\ell,2)$ convergent to s and $n\log n\log(\log n)\Delta u_n=o(1)$, then (u_n) converges to the same value.

Proof. We need to show that the condition $n\log n\log(\log n)\Delta u_n=o(1)$ implies $V_{n,2}(\Delta u)=o(1)$ as $n\to\infty$. In fact, for a sequence (u_n), since we take $P_n=\gamma_{k,2}$ and $m=0$ in Totur & Çanak (2012, Lemma 1), we obtain the identity

$$u_n-\ell_{n,2}(u)=V_{n,2}(\Delta u). \tag{8}$$

Since $n\log n\log(\log n)\Delta u_n=o(1)$, the condition $V_{n,2}(\Delta u)=o(1)$ as $n\to\infty$ is satisfied from the regularity of $(\ell,2)$ summability method.

Notice that the Tauberian condition in Theorem 5 can not be replaced with the conditions $V_{n,2}(\Delta u)=O(1)$ or $V_{n,2}(\Delta u)\ge -C$, for some $C>0$. Although the following one-sided condition is a Tauberian condition for the $(\ell,2)$ summability method.

Theorem 7. If (u_n) is $(\ell,2)$ convergent to s and

$$n\log n\log(\log n)\Delta V_{n,2}(\Delta u)\ge -C,$$

for some $C>0$, then (u_n) converges to the same value.

It is clear that the Tauberian condition of Theorem 7 is more general than the condition in Corollary 6. Indeed, if we write the term $n\log n\log(\log n)\Delta u_n$ instead of u_n in the identity (7), then the identity

$$n\gamma_{k,1}\gamma_{k,2}\Delta u_n-V_{n,2}(\Delta u)=n\gamma_{k,1}\gamma_{k,2}\Delta V_{n,2}(\Delta u)$$

is obtained by the identity (8).

Therefore, the condition $n\log n\log(\log n)\Delta u_n=o(1)$ implies $n\log n\log(\log n)\Delta V_{n,2}(\Delta u)=o(1)$ as $n\to\infty$.

Corollary 8. If (u_n) is $(\ell,2)$ convergent to s and $n\log n\log(\log n)\Delta V_{n,2}(\Delta u)=O(1)$, then (u_n) converges to the same value.

Now, we give a Tauberian theorem which generalize Theorem 7. In the next theorem, we show that we obtain the convergence of the sequence (u_n) under the condition of slow decrease of the difference of (u_n) and its $(\ell,2)$ mean sequence $(\ell_{n,2}(u))$ instead of being the slow decrease of (u_n).

Theorem 9. If (u_n) is $(\ell,2)$ convergent to s and $(V_{n,2}(\Delta u))$ is slowly decreasing with respect to $(\ell,2)$, then (u_n) converges to the same value.

A natural corollary of Theorem 9 is given as follows.

Corollary 10. If (u_n) is $(\ell,2)$ convergent to s and $(V_{n,2}(\Delta u))$ is slowly oscillating with respect to $(\ell,2)$, then (u_n) converges to the same value.

In the last theorem, we extend Móricz's theorem (Móricz, 2013) to $(\ell,2)$ summability method.

Theorem 11. If (u_n) is $(\ell,2)$ convergent to s and the conditions

$$\limsup_{\lambda\to 1^+}\liminf_{n\to\infty}\frac{1}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^\lambda}]}\frac{u_k-u_n}{(k+1)\gamma_{k,1}}\geq 0,$$

$$\limsup_{\lambda\to 1^-}\liminf_{n\to\infty}\frac{1}{\gamma_{n,2}-\gamma_{[e^{(\log n)^\lambda}],2}}\sum_{k=[e^{(\log n)^\lambda}]+1}^{n}\frac{u_n-u_k}{(k+1)\gamma_{k,1}}\geq 0,$$

are satisfied, then (u_n) converges to the same value.

Theorem 11 can be proved by using the same steps as that Theorem 2.1 in (Móricz, 2013). So, the proof of Theorem 11 is omitted.

3. A Lemma

In the proofs of main theorems, the following lemma will be used.

Lemma 12. (i) For $\lambda>1$,

$$u_n-\ell_{n,2}(u)=\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^\lambda}],2}(u)-\ell_{n,2}(u))-\frac{1}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^\lambda}]}\frac{u_k-u_n}{(k+1)\gamma_{k,1}}.$$

(ii) For $0<\lambda<1$,

$$u_n-\ell_{n,2}(u)=\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{n,2}-\gamma_{[e^{(\log n)^\lambda}],2}}(\ell_{n,2}(u)-\ell_{[e^{(\log n)^\lambda}],2}(u))+\frac{1}{\gamma_{n,2}-\gamma_{[e^{(\log n)^\lambda}],2}}\sum_{k=[e^{(\log n)^\lambda}]+1}^{n}\frac{u_n-u_k}{(k+1)\gamma_{k,1}}.$$

Proof. (i) From the definition of the logarithmic de la Vallée Poussin mean of (u_n) with respect to $(\ell,2)$, we have

$$\tau^{>}_{[e^{(\log n)^\lambda}],2}(u)=\frac{1}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}\left(\sum_{k=0}^{[e^{(\log n)^\lambda}]}\frac{u_k}{(k+1)\gamma_{k,1}}-\sum_{k=0}^{n}\frac{u_k}{(k+1)\gamma_{k,1}}\right)$$

$$=\frac{1}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}(\gamma_{[e^{(\log n)^\lambda}]}\,\ell_{[e^{(\log n)^\lambda}],2}(u)-\gamma_{n,2}\ell_{n,2}(u))$$

$$=\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}\ell_{[e^{(\log n)^\lambda}],2}(u)-\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}\ell_{n,2}(u)+\ell_{n,2}(u)$$

$$=\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^\lambda}],2}(u)-\ell_{n,2}(u))+\ell_{n,2}(u).$$

Substracting $\ell_{n,2}(u)$ from the last identity, we get

$$\tau^{>}_{[e^{(\log n)^{\lambda}}],2}(u)-\ell_{n,2}(u)=\frac{\gamma_{[e^{(\log n)^{\lambda}}],2}}{\gamma_{[[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^{\lambda}}],2}(u)-\ell_{n,2}(u)). \tag{9}$$

Also the sequence (u_n) can be written as

$$u_n=\frac{1}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^{\lambda}}]}\frac{u_k}{(k+1)\gamma_{k,1}}-\frac{1}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^{\lambda}}]}\frac{1}{(k+1)\gamma_{k,1}}(u_k-u_n). \tag{10}$$

Substracting $\ell_{n,2}(u)$ from the identity (10), we obtain

$$u_n-\ell_{n,2}(u)=\tau^{>}_{[e^{(\log n)^{\lambda}}],2}(u)-\ell_{n,2}(u)-\frac{1}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^{\lambda}}]}\frac{1}{(k+1)\gamma_{k,1}}(u_k-u_n).$$

Writing (9) in the last identity, we obtain

$$u_n-\ell_{n,2}(u)=\frac{\gamma_{[e^{(\log n)^{\lambda}}],2}}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^{\lambda}}],2}(u)-\ell_{n,2}(u))-\frac{1}{\gamma_{[e^{(\log n)^{\lambda}}]}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^{\lambda}}]}\frac{1}{(k+1)\gamma_{k,1}}(u_k-u_n).$$

This completes the proof.

(ii) The proof of Lemma 12 (ii) is similar to that of Lemma 12 (i).

4. Main results for $(\ell,1)$ summability method

All the main theorems established above can be given in the form below according to the $(\ell,1)$ summability method. We omit the proof of the theorems since the theorems are proved with similar steps.

Theorem 13. If (u_n) is $(\ell,1)$ convergent to s and $n\log n\Delta V_{n,1}(\Delta u)\geq -C$, for some $C>0$, where $V_{n,1}(\Delta u)=\frac{1}{\gamma_{n,1}}\sum_{k=1}^{n}\gamma_{k-1,1}\Delta u_k$, then (u_n) is convergent to s.

Corollary 14. If (u_n) is $(\ell,1)$ convergent to s and $n\log n\Delta V_{n,1}(\Delta u)=O(1)$, then (u_n) is convergent to s.

Theorem 15. If (u_n) is $(\ell,1)$ convergent to s and the sequence $(V_{n,1}(\Delta u))$ is slowly decreasing with respect to $(\ell,1)$, then (u_n) is convergent to s.

Corollary 16. If (u_n) is $(\ell,1)$ convergent to s and the sequence $(V_{n,1}(\Delta u))$ is slowly oscillating with respect to $(\ell,1)$, then (u_n) is convergent to s.

We require the following lemma to be used in the proofs of main theorems.

Lemma 17. (i) For $\lambda>1$,

$$u_n-\ell_{n,1}(u)=\frac{\gamma_{[n^{\lambda}]}}{\gamma_{[n^{\lambda}]}-\gamma_n}(\ell_{[n^{\lambda}],1}(u)-\ell_{n,1}(u))-\frac{1}{\gamma_{[n^{\lambda}]}-\gamma_n}\sum_{k=n+1}^{[n^{\lambda}]}\frac{u_k-u_n}{k+1}.$$

(ii) For $0<\lambda<1$,

$$u_n-\ell_{n,1}(u)=\frac{\gamma_{[n^{\lambda}]}}{\gamma_n-\gamma_{[n^{\lambda}]}}(\ell_{n,1}(u)-\ell_{[n^{\lambda}],1}(u))+\frac{1}{\gamma_n-\gamma_{[n^{\lambda}]}}\sum_{k=[n^{\lambda}]+1}^{n}\frac{u_n-u_k}{k+1}.$$

5. Proofs

In this section, we give the proof of the main theorems of the paper which are Theorem 7 and Theorem 9 for the logarithmic summability method of order 2.

Proof of Theorem 7

Suppose that $n\log n\log(\log n)\Delta V_{n,2}(\Delta u)\geq -C$ for some $C\geq 0$. Then, we obtain

$$-\Delta V_{n,2}(\Delta u)\leq \frac{C}{n\log n\log(\log n)}.$$

From Lemma 12 (i), we have

$$V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u))=\frac{\gamma_{[e^{(\log n)^{\lambda}}],2}}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^{\lambda}}],2}(V_2(\Delta u))-\ell_{n,2}(V_2(\Delta u))$$

$$-\frac{1}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^{\lambda}}]}\frac{1}{(k+1)\gamma_{k,1}}(V_{k,2}(\Delta u)-V_{n,2}(\Delta u))$$

$$=\frac{\gamma_{[e^{(\log n)^{\lambda}}],2}}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^{\lambda}}],2}(V_2(\Delta u))-\ell_{n,2}(V_2(\Delta u))$$

$$-\frac{1}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^{\lambda}}]}\frac{1}{(k+1)\gamma_{k,1}}\sum_{j=n+1}^{k}\Delta V_{j,2}(\Delta u)$$

$$\leq\frac{\gamma_{[e^{(\log n)^{\lambda}}],2}}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^{\lambda}}],2}(V_2(\Delta u))-\ell_{n,2}(V_2(\Delta u))$$

$$+\frac{1}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^{\lambda}}]}\frac{1}{(k+1)\gamma_{k,1}}\sum_{j=n+1}^{k}\frac{c}{j\log j\log(\log j)}$$

$$\leq\frac{\gamma_{[e^{(\log n)^{\lambda}}],2}}{\gamma_{[e^{(\log n)^{\lambda}}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^{\lambda}}],2}(V_2(\Delta u))-\ell_{n,2}(V_2(\Delta u))$$

$$+C_1\log\left(\frac{\log(\log[e^{(\log n)^{\lambda}}])}{\log(\log n)}\right),$$

for some $C_1 > 0$. Taking limsup of both sides of the inequality above as $n \to \infty$, and using that

$$\limsup_{n\to\infty} \frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{[e^{(\log n)^\lambda}],2} - \gamma_{n,2}} \leq \frac{\lambda}{\lambda - 1},$$

we get

$$\limsup_{n\to\infty}(V_{n,2}(\Delta u) - \ell_{n,2}(V_2(\Delta u))) \leq \frac{\lambda}{\lambda-1}(\ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u)) - \ell_{n,2}(V_2(\Delta u))) + C_1 \log \lambda.$$

Since (u_n) is logarithmic summable of order 2 to s, we have $(\ell_{n,2}(u))$ is convergent to s. Because of regularity of the $(\ell, 2)$ summability method, $(\ell_{n,2}(u))$ is $(\ell, 2)$ convergent to s. Therefore, $(\ell_{n,2}(V_2(\Delta u)))$ is convergent to 0.

Hence the first term on the right-hand side of the inequality above vanishes and we obtain

$$\limsup_{n\to\infty}(V_{n,2}(\Delta u) - \ell_{n,2}(V_2(\Delta u))) \leq C_1 \log \lambda.$$

After taking the limit of both sides as $\lambda \to 1^+$, we get

$$\limsup_{n\to\infty}(V_{n,2}(\Delta u) - \ell_{n,2}(V_2(\Delta u))) \leq 0. \tag{11}$$

On the other hand, from Lemma 12 (ii) and the hypothesis $n \log n \log(\log n) \Delta V_{n,2}(\Delta u) \geq -C$ for some $C > 0$, we have

$$V_{n,2}(\Delta u) - \ell_{n,2}(V_2(\Delta u)) = \frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{n,2} - \gamma_{[e^{(\log n)^\lambda}],2}}(\ell_{n,2}(V_2(\Delta u)) - \ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u)))$$

$$+ \frac{1}{\gamma_{n,2} - \gamma_{[e^{(\log n)^\lambda}],2}} \sum_{k=[e^{(\log n)^\lambda}]+1}^{n} \frac{1}{(k+1)\gamma_{k,1}}(V_{n,2}(\Delta u) - V_{k,2}(\Delta u))$$

$$= \frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{n,2} - \gamma_{[e^{(\log n)^\lambda}],2}}(\ell_{n,2}(V_2(\Delta u)) - \ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u)))$$

$$+ \frac{1}{\gamma_{[e^{(\log n)^\lambda}],2} - \gamma_{n,2}} \sum_{k=[e^{(\log n)^\lambda}]+1}^{n} \frac{1}{(k+1)\gamma_{k,1}} \sum_{j=n+1}^{k} \Delta V_{j,2}(\Delta u)$$

$$\geq \frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{n,2} - \gamma_{[e^{(\log n)^\lambda}],2}}(\ell_{n,2}(V_2(\Delta u)) - \ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u)))$$

$$- \frac{1}{\gamma_{[e^{(\log n)^\lambda}],2} - \gamma_{n,2}} \sum_{k=[e^{(\log n)^\lambda}]+1}^{n} \frac{1}{(k+1)\gamma_{k,1}} \sum_{j=n+1}^{k} \frac{C}{j \log j \log(\log j)}$$

$$\geq \frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{n,2} - \gamma_{[e^{(\log n)^\lambda}],2}}(\ell_{n,2}(V_2(\Delta u)) - \ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u)))$$

$$-C_1 \log\left(\frac{\log(\log[e^{(\log n)^\lambda}])}{\log(\log n)}\right),$$

for some $C_1 > 0$. After taking liminf of both sides as $n\to\infty$, and using that

$$\liminf_{n\to\infty}\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{n,2}-\gamma_{[e^{(\log n)^\lambda}],2}}\geq\frac{\lambda}{1-\lambda},$$

we get

$$\liminf_{n\to\infty}(V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u)))\geq\frac{\lambda}{1-\lambda}(\ell_{n,2}(V_2(\Delta u))-\ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u)))-C_1\log\lambda.$$

Since $(\ell_{n,2}(V_2(\Delta u)))$ is convergent to 0, the first term on the right-hand side of the inequality above vanishes and we obtain

$$\liminf_{n\to\infty}(V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u)))\geq C_1\log\lambda.$$

Taking the limit of both sides as $\lambda\to 1^-$, we get

$$\liminf_{n\to\infty}(V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u)))\geq 0. \tag{12}$$

Combining (11) and (12), we obtain $\lim_{n\to\infty}V_{n,2}(\Delta u)=\lim_{n\to\infty}\ell_{n,2}(V_2(\Delta u))=0$. By Theorem 5, we obtain $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\ell_{n,2}(u)$.

Proof of Theorem 9

From Lemma 12 (i), we have

$$V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u))=\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u))-\ell_{n,2}(V_2(\Delta u)))$$

$$-\frac{1}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^\lambda}]}\frac{1}{(k+1)\gamma_{k,1}}(V_{k,2}(\Delta u)-V_{n,2}(\Delta u)),$$

and

$$V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u))\leq\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u))-\ell_{n,2}(V_2(\Delta u)))$$

$$-\frac{1}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}\sum_{k=n+1}^{[e^{(\log n)^\lambda}]}\frac{1}{(k+1)\gamma_{k,1}}\min_{n<k\leq e^{(\log n)^\lambda}}(V_{k,2}(\Delta u)-V_{n,2}(\Delta u))$$

$$\leq\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u))-\ell_{n,2}(V_2(\Delta u)))$$

$$-\min_{n<k\leq e^{(\log n)^\lambda}}(V_{k,2}(\Delta u)-V_{n,2}(\Delta u)).$$

After taking limsup of both sides as $n \to \infty$, we have

$$\limsup_{n\to\infty}\left(V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u))\right)\le \limsup_{n\to\infty}\left(\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u))-\ell_{n,2}(V_2(\Delta u))) - \min_{n<k\le e^{(\log n)^\lambda}}(V_{k,2}(\Delta u)-V_{n,2}(\Delta u))\right)$$

$$\le \limsup_{n\to\infty}\left(\frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{[e^{(\log n)^\lambda}],2}-\gamma_{n,2}}(\ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u))-\ell_{n,2}(V_2(\Delta u)))\right) + \limsup_{n\to\infty}\left(-\min_{n<k\le e^{(\log n)^\lambda}}(V_{k,2}(\Delta u)-V_{n,2}(\Delta u))\right)$$

$$\le \frac{\lambda}{\lambda-1}\limsup_{n\to\infty}\left(\ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u))-\ell_{n,2}(V_2(\Delta u))\right) + \limsup_{n\to\infty}\left(-\min_{n<k\le e^{(\log n)^\lambda}}(V_{k,2}(\Delta u)-V_{n,2}(\Delta u))\right).$$

Since $(\ell_{n,2}(u))$ converges to s, the first term on the right-hand side of the equality above vanishes and we obtain

$$\limsup_{n\to\infty}\left(V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u))\right)\le -\liminf_{n\to\infty}\min_{n<k\le e^{(\log n)^\lambda}}(V_{k,2}(\Delta u)-V_{n,2}(\Delta u)).$$

After taking the limit of both sides as $\lambda \to 1^+$, we get

$$\liminf_{n\to\infty}(V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u)))\ge 0. \tag{13}$$

On the other hand, from Lemma 12 (ii), we have

$$V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u)) = \frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{n,2}-\gamma_{[e^{(\log n)^\lambda}],2}}(\ell_{n,2}(V_2(\Delta u))-\ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u))) + \frac{1}{\gamma_{n,2}-\gamma_{[e^{(\log n)^\lambda}],2}}\sum_{k=[e^{(\log n)^\lambda}]+1}^{n}\frac{1}{(k+1)\gamma_{k,1}}(V_{n,2}(\Delta u)-V_{k,2}(\Delta u))$$

$$\ge \frac{\gamma_{[e^{(\log n)^\lambda}],2}}{\gamma_{n,2}-\gamma_{[e^{(\log n)^\lambda}],2}}(\ell_{n,2}(V_2(\Delta u))-\ell_{[e^{(\log n)^\lambda}],2}(V_2(\Delta u)))$$

$$+\frac{1}{\gamma_{n,2}-\gamma_{[e^{(\log n)^{\lambda}}],2}}\sum_{k=[e^{(\log n)^{\lambda}}]+1}^{n}\frac{1}{(k+1)\gamma_{k,1}}\min_{e^{(\log n)^{\lambda}}<k\leq n}(V_{n,2}(\Delta u)$$

$$-V_{k,2}(\Delta u))$$

$$\geq\frac{\gamma_{[e^{(\log n)^{\lambda}}],2}}{\gamma_{n,2}-\gamma_{[e^{(\log n)^{\lambda}}],2}}(\ell_{n,2}(V_2(\Delta u))-\ell_{[e^{(\log n)^{\lambda}}],2}(V_2(\Delta u)))$$

$$+\min_{e^{(\log n)^{\lambda}}<k\leq n}(V_{n,2}(\Delta u)-V_{k,2}(\Delta u)).$$

After taking liminf of both sides as $n\to\infty$, we get

$$\liminf_{n\to\infty}\left(V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u))\right)\geq\liminf_{n\to\infty}\left(\frac{\gamma_{[e^{(\log n)^{\lambda}}],2}}{\gamma_{n,2}-\gamma_{[e^{(\log n)^{\lambda}}],2}}(\ell_{n,2}(V_2(\Delta u))-\ell_{[e^{(\log n)^{\lambda}}],2}(V_2(\Delta u)))\right.$$

$$\left.+\min_{e^{(\log n)^{\lambda}}<k\leq n}(V_{n,2}(\Delta u)-V_{k,2}(\Delta u))\right)$$

$$\geq\liminf_{n\to\infty}\left(\frac{\gamma_{[e^{(\log n)^{\lambda}}],2}}{\gamma_{n,2}-\gamma_{[e^{(\log n)^{\lambda}}],2}}(\ell_{n,2}(V_2(\Delta u))-\ell_{[e^{(\log n)^{\lambda}}],2}(V_2(\Delta u)))\right)$$

$$+\liminf_{n\to\infty}\left(\min_{e^{(\log n)^{\lambda}}<k\leq n}(V_{n,2}(\Delta u)-V_{k,2}(\Delta u))\right)$$

$$\geq\frac{\lambda}{1-\lambda}\liminf_{n\to\infty}\left((\ell_{n,2}(V_2(\Delta u))-\ell_{[e^{(\log n)^{\lambda}}],2}(V_2(\Delta u)))\right)$$

$$+\liminf_{n\to\infty}\left(\min_{e^{(\log n)^{\lambda}}<k\leq n}(V_{n,2}(\Delta u)-V_{k,2}(\Delta u))\right).$$

Since $(\ell_{n,2}(u))$ converges to s, the first term on the right-hand side of the equality above vanishes and we obtain

$$\liminf_{n\to\infty}\left(V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u))\right)\geq\liminf_{n\to\infty}\min_{e^{(\log n)^{\lambda}}<k\leq n}(V_{n,2}(\Delta u)-V_{k,2}(\Delta u)).$$

After taking the limit of both sides as $\lambda\to1^{-}$, we get

$$\liminf_{n\to\infty}\left(V_{n,2}(\Delta u)-\ell_{n,2}(V_2(\Delta u))\right)\geq0. \tag{14}$$

Combining (13) and (14), we have $\lim_{n\to\infty}V_{n,2}(\Delta u)=\lim_{n\to\infty}\ell_{n,2}(V_2(\Delta u))$. Hence, we obtain that $\lim_{n\to\infty}u_n=s$.

6. Conclusion

A convergent sequence is both logarithmic summable of order 1 and 2 to the same limit. But the converse of the both statements are not true in general. Our main results in the present study answer the question under which conditions logarithmic summable sequences of order 1 or 2 are convergent, and improve some classical Tauberian theorems.

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[1,*]إميت توتر، [2]محمد أوكر
[1]قسم الرياضيات، جامعة عدنان مندريس، 09010، أيدين، تركيا
[2]قسم الرياضيات، جامعة عدنان مندريس، 09010، أيدين، تركيا
المؤلف: utotur@adu.edu.tr, utotur@yahoo.com

خلاصة

ندرس في هذا البحث طرق الجمع للمعدلات اللوغاريتمية للمتتاليات العددية و تطبيقاتها للحصول على مبرهنات من النوع التوبارباني.