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Simultaneous approximation by polynomials in Orlicz spaces generated by quasiconvex Young functions

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Abstract

In this paper we prove some theorems on simultaneous approximation by trigonometric or algebraic polynomials in Orlicz spaces constructed by Young functions belonging to a reasonably wide class.

Keywords: Moduli of smoothness; Orlicz spaces; simultaneous approximations; trigonometric/algebraic approximation.

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1. Introduction

The problems of approximation by trigonometric/algebraic polynomials in classical Orlicz spaces were investigated by several mathematicians. Tsyganok (1966) obtained the Jackson type inequality of trigonometric approximation. Kokilashvili (1965) obtained inverse theorems of trigonometric approximation. Ponomarenko (1966) proved some direct theorem of trigonometric approximation by summation means of Fourier series. Cohen (1968) proved some direct theorem of trigonometric approximation by its partial sum of Fourier series. On the other hand simultaneous approximation of functions by trigonometric/algebraic polynomials in the classical Orlicz spaces were proved by Ramazanov (1984) and Garidi (1991). In these results the generating Young function of Orlicz spaces is convex. When the generating Young function satisfying quasiconvexity condition, similar problems were investigated in Akgün (2012), Akgün (2011), Akgün (2016), Akgün & Israfilov (2011) and Israfilov & Akgün (2010).

Present work deals with central problems of approximation by trigonometric/algebraic polynomials in Orlicz spaces having non convex generating Young functions. First of all we give basic definitions and notations.

Let $\varphi:[0,\infty)\to[0,\infty)$ be a right continuous, monotone increasing function with $f(0)=0$; $\lim\limits_{t\to\infty} f(t)=\infty$ and $f(t)>0$ whenever $t>0$; then the function defined by

$$\Phi(x)=\int_0^{|x|} f(t)dt$$

is called N-function (Krasnosel'skii & Rutickii, 1961). The class of strictly increasing functions will be denoted by Φ. When φ is an N-function (Krasnosel'skii & Rutickii, 1961) we always denote by $\psi(u)$ the mutually complementary N-function of $\varphi(u)$. Let $\varphi(u)$ be an N-function. We shall denote by L_φ the class of real-valued functions, defined on $I:=[a,b]\subset R$ such that

$$\rho(u;\varphi):=\int_a^b \varphi\left[\left|u(x)\right|\right]dx<\infty.$$

The classes L_φ are called Orlicz classes. The class of measurable functions f defined on I such that the product $f(x)g(x)$ is integrable over (a,b) for every measurable function $g\in L_\Psi$, will be denoted by L_φ^* which is called (classical) Orlicz space. We put

$$\|f\|:=\|f\|_\varphi:=\sup_g\left|\int_a^b f(x)g(x)dx\right|$$

where the supremum being taken with respect to all g with

$$\rho_g:=\rho(g,\psi)=\int_a^b \psi(|g(x)|)dx\le 1.$$

Now we will give definition of another class of functions (Chen, 1964) that is wider than the classical Orlicz spaces L_φ^*. The wider class will be denoted by L_φ^{**} and the generating function φ of L_φ^{**} is not necessary to be convex (Chen, 1964). We set $-\infty<p\le q<\infty$ and we denote by $Y[p,q]$ the class of even functions $\varphi \in \Phi$ defined on $(-\infty,\infty)$ satisfying the following two conditions

(i) $\varphi(u)/u^p$ is non-decreasing when $|u|$ increases;

(ii) $\varphi(u)/u^q$ is non-increasing when $|u|$ increases.

When $p<q$ we will denote by $Y\langle p,q\rangle$ the class of functions φ satisfying $\varphi \in Y[p+\varepsilon, q-\delta]$ for some small numbers $\varepsilon,\delta>0$.

The notation $\varphi(x)\sim[p_1,p_2]$, $0\le p_1\le p_2\le\infty$ (or similarly $-\infty\le p_1\le p_2\le 0$) will indicate that for non-negative even function $\varphi(x)$, the function $\varphi(x)\cdot x^{-p_1}$ is non-decreasing and the function $\varphi(x)\cdot x^{-p_2}$ is non-increasing when x is increasing in $(0,\infty)$.

Let $\varphi(x)\sim[p_1,p_2], 0\le p_1\le p_2\le\infty$ and $\varphi_1(t):=\varphi(t)/t$. We suppose that $\varphi_1(t)\to\infty$ as $t\to\infty$, and $\psi_1(t)$ be the inverse function of the positive non-decreasing continuous function φ_1. Defining as

$$\Phi_1(x):=\int_0^x \varphi_1(t)dt \text{ and } \Psi_1(x):=\int_0^x \psi_1(t)dt$$

we get that Φ_1 is a convex function and the functions Φ_1, Ψ_1 are complementary functions. By L_φ^{**} we will define the set of functions $f(x)$, $a\le x\le b$, such that the product $f(x)$ $g(x)$ is integrable over (a,b) for any $g\in L_{\Psi_1}$. In L_φ^{**} one can define a norm as

$$\|f\|_{\langle\varphi\rangle} := \|f\|_{\langle\varphi\rangle(I)} := \sup_g \left|\int_I f(x)g(x)dx\right|, \quad (1.1)$$

where the supremun being taken for all g satisfying

$$\rho(g,\Psi_1)=\int_a^b \Psi_1(|g(x)|)dx\le 1. \quad (1.2)$$

Namely

$$L_\varphi^{**}:=\left\{f:[a,b]\to R \mid \int_a^b f(x)g(x)dx<\infty \text{ for all } g\in L_{\Psi_1}\right\}.$$

For $f\in L_\varphi^{**}$, $g\in L_{\Psi_1}^{*}$ the generalized Hölder inequality (Chen, 1964)

$$\left|\int_a^b f(x)g(x)dx\right| \le \|f\|_{\langle\varphi\rangle}\|g\|_{(\Psi_1)} \quad (1.3)$$

holds, where

$$\|g\|_{(\Psi_1)} := \inf\left\{k>0:\rho\left(\frac{g(x)}{k};\Psi_1\right)\le 1\right\}.$$

There are several important classes of $N-$functions. Among other things, these conditions relate to the growth of $N-$functions. Let φ be an $N-$function. Then φ is said to satisfy the Δ_2 doubling condition (in notation: $\varphi\in\Delta_2$) (Chen, 1964); namely, there is a constant $C>0$ and $x_0>0$ such that $\varphi(2x)\le C\varphi(x)$ for all $x\ge x_0$. An Orlicz class is linear if and only if it satisfies the Δ_2 doubling condition (Krasnosel'skii & Rutickii, 1961). Then the above defined function classes L_φ, L_φ^{*} and L_φ^{**} are identical (Chen, 1964). In general the class L_φ^{**} is wider than L_φ^{*} and L_φ^{**} preserves the same properties of L_φ^{*}. In the class L_φ^{**} the function φ is not necessary to satisfy the convexity condition.

Ramazanov (1984) has obtained Jackson type theorem for the functions in Orlicz spaces L_φ^{*}. For further results see e.g. Akgün (2012), Akgün (2011), Akgün & Koç (2012) and Akgün & Koç (2016). Later Garidi (1991) extended the results of Ramazanov and proved Jackson type theorem for derivatives in the space

$$L_\varphi^{*,r}:=\left\{f\in L_\varphi^{*} \mid f^{(r)}\in L_\varphi^{*}\right\}.$$

But there are functions in L_φ^{**} or $L_{\varphi,\pi}^{**}$ that does not belong to the class L_φ^{*} or $L_{\varphi,\pi}^{*}$.

For example taking as $\varphi\sim\langle 2,3\rangle$, $\varphi(x)=x^{5/2}$ for $0\le x\le 1$, $\varphi(x)=x^{9/4}$ for $x>1$ we have that (Chen, 1964) φ is not a convex function. There exists (Chen, 1964) a method to find such functions. The main aim of this paper is to consider the simultaneous approximation by algebraic/trigonometric polynomials for functions in the Sobolev type space.

For $a=-\pi$ and $b=\pi$ then we will use the notations $L_{\varphi,\pi}^{**}$ and $L_{\varphi,\pi}^{**,r}$. For the simplicity every where in this work, the constant c will denote different positive real number in different places.

$$L_\varphi^{**,r}:=\left\{f:f,f^{(r)}\in L_\varphi^{**}\setminus L_\varphi^{*}\right\},$$

$$L_{\varphi,\pi}^{**,r}:=\left\{f:f,f^{(r)}\in L_{\varphi,\pi}^{**}\setminus L_{\varphi,\pi}^{*}\right\}.$$

Our main results are the following.

Theorem 1.1. *Let* $1<p<q<\infty$, $\varphi\in Y\langle p,q\rangle$, $r=1,2,3,\ldots$ *and* $\upsilon=0,1,2,\ldots,r$. *For any* $f\in L_\varphi^{**,r}$ *there exists an algebraic polynomial P of degree n such that*

$$\left\|f^{(\upsilon)}-P^{(\upsilon)}\right\|_{\langle\varphi\rangle}\le c\,\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},1/n\right)$$

holds for any integer $n\ge 1$ *where c is some constant depending only on r and* φ.

Theorem 1.2. *Let* $1<p<q<\infty$, $\varphi\in Y\langle p,q\rangle$, $r=1,2,3,\ldots$ *and* $\upsilon=0,1,2,\ldots,r$. *For any* $f\in L_{\varphi,\pi}^{**,r}$ *there exists a trigonometric polynomial T of degree n such that*

$$\left\|f^{(\upsilon)}-T^{(\upsilon)}\right\|_{\langle\varphi\rangle}\le c\,\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},1/n\right)$$

holds for any integer $n\ge 1$ *where c is some constant depending only on r and* φ.

In these theorems $\omega_{r,\langle\varphi\rangle}$ denotes r-th modulus of smoothness which given in (2.3).

2. K – Functional and modulus of smoothness in L_{φ}^{**}

Suppose that $r=1,2,3,...$, $\upsilon=0,1,2,...,r$, $t>0$ and $f\in L_{\varphi}^{**,r}$. We define

$$\|f\|_{r,\langle\varphi\rangle,t}=\|f\|_{r,\langle\varphi\rangle}:=\sum_{i=0}^{\upsilon}t^{i}\left\|f^{(i)}\right\|_{\langle\varphi\rangle} \tag{2.1}$$

and K-functional by

$$K_{r,\varphi}^{\upsilon}(f,t):=\inf\left\{\|f-g\|_{\upsilon,\langle\varphi\rangle}+t^{r}\left\|g^{(r)}\right\|_{\upsilon,\langle\varphi\rangle}:g\in L_{\varphi}^{**,(r+\upsilon)}\right\}. \tag{2.2}$$

For $h\ge 0$ and

$$I_h:=\begin{cases}[a,b-h] & ,0\le h\le b-a\\ \varnothing & ,h>b-a\end{cases},$$

the expression

$$\Delta_t^{r}(f,x):=\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}f(x+it)$$

is called r-th difference of the function f.

For the empty set $\varnothing$, we define $\|.\|_{\langle\varphi\rangle(\varnothing)}=0$. For $f\in L_{\varphi}^{**}$, we define its r-th modulus of smoothness as

$$\omega_{r,\langle\varphi\rangle}(f,t)=\sup_{0\le h\le t}\left\|\Delta_h^{r}(f,.)\right\|_{\langle\varphi\rangle(I_{rh})}. \tag{2.3}$$

Remark 2.1. *The modulus* $\omega_{r,\langle\varphi\rangle}(f,t)$ *possesses the following usual properties:*

(1) $\omega_{r,\langle\varphi\rangle}(f,t)$ *is a monotone increasing function of t and* $\omega_{r,\langle\varphi\rangle}(f,0)=0$.

(2) *For any* $f\in L_{\varphi}^{**}$ $\omega_{r,\langle\varphi\rangle}(f,t)\to 0$ *as* $t\to 0$ *iff* φ *function sastisfies* Δ_2*-condition.*

(3) *If* $f\in L_{\varphi}^{**,r}$, *then* $\omega_{r+n,\langle\varphi\rangle}(f,t)\le t^{n}\omega_{r,\langle\varphi\rangle}\left(f^{(n)},t\right)$.

(4) $\omega_{r,\langle\varphi\rangle}(f,nt)\le n^{r}\omega_{r,\langle\varphi\rangle}(f,t)$ *for any non-negative integer n.*

Theorem 2.2. *The function* φ *satisfies the* Δ_2*-condition if and only if "for given any positive integer r,* $\omega_{r,\langle\varphi\rangle}(f,t)\to 0$ *holds true as* $t\to 0$ *for every* $f\in L_{\varphi}^{**}$*."*

Proof of Theorem 2.2. Suppose that φ satisfies the Δ_2-condition and that P_n $(n=1,2,...)$ is a sequence of polynomials which convergence to $f\in L_{\varphi}^{**}$. As in Garidi (1991) for $0<h<\frac{b-a}{r}$ we have

$$\left\|\Delta_r^{h}(f-P_n,x)\right\|_{\langle\varphi\rangle[a,b-rh]}=\sup_{g\in L_{\Psi_1}}\left\{\int_a^{b-rh}\left|\Delta_r^{h}(f-P_n,x)g(x)\right|dx:\rho(g,\Psi_1)\le 1\right\}$$

$$=\sup\left\{\int_a^{b-rh}\left|\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(f-P_n)(x+ih)\right|\left|g(x)\right|dx:g\in L_{\Psi_1},\rho(g,\Psi_1)\le 1\right\}$$

$$\le\sup\left\{\int_a^{b-rh}\sum_{i=0}^{r}\binom{r}{i}\left|f(x+ih)-P_n(x+ih)\right|\left|g(x)\right|dx:g\in L_{\Psi_1},\rho(g,\Psi_1)\le 1\right\}$$

$$=\sup\left\{\sum_{i=0}^{r}\int_a^{b-rh}\binom{r}{i}\left|f(x+ih)-P_n(x+ih)\right|\left|g(x)\right|dx:g\in L_{\Psi_1},\rho(g,\Psi_1)\le 1\right\}$$

$$=\sup\left\{\sum_{i=0}^{r}\binom{r}{i}\int_a^{b-rh}\left|f(x+ih)-P_n(x+ih)\right|\left|g(x)\right|dx:g\in L_{\Psi_1},\rho(g,\Psi_1)\le 1\right\}$$

$$\le\sum_{i=0}^{r}\binom{r}{i}\left\{\sup\int_a^{b-rh}\left|f(x+ih)-P_n(x+ih)\right|\left|g(x)\right|dx:g\in L_{\Psi_1},\rho(g,\Psi_1)\le 1\right\}$$

$$=\sum_{i=0}^{r}\binom{r}{i}\left\|f(x+ih)-P_n(x+ih)\right\|_{\langle\varphi\rangle[a,b-rh]}.$$

Since the space is invariant under translation, writing $x+ih=u$ and $0\le i\le r$, $a\le x\le br-h$, $a\le x+rh\le b$ we obtain

$$\begin{aligned}\left\|\Delta_r^h(f,\cdot)\right\|_{\langle\varphi\rangle[a,b-rh]} &= \left\|\Delta_r^h(f-P_n,\cdot)+\Delta_r^h(P_n,\cdot)\right\|_{\langle\varphi\rangle[a,b-rh]}\\ &\le \left\|\Delta_r^h(f-P_n,\cdot)\right\|_{\langle\varphi\rangle[a,b-rh]}+\left\|\Delta_r^h(P_n,\cdot)\right\|_{\langle\varphi\rangle[a,b-rh]}\\ &\le 2^r\left\|f-P_n\right\|_{\langle\varphi\rangle[a,b-rh]}+\left\|\Delta_r^h(P_n,\cdot)\right\|_{\langle\varphi\rangle[a,b-rh]}.\end{aligned}$$

Therefore we have

$$\left\|\Delta_r^h(f,\cdot)\right\|_{\langle\varphi\rangle[a,b-rh]}\le 2^r\left\|f-P_n\right\|_{\langle\varphi\rangle[a,b-rh]}+\left\|\Delta_r^h(P_n,\cdot)\right\|_{\langle\varphi\rangle[a,b-rh]}<\varepsilon.$$

Thus, $\omega_{r,\langle\varphi\rangle}\to 0$, as $n\to\infty$ since

$$c\omega_{r,\langle\varphi\rangle}(f)<\varepsilon.$$

The opposite way of Theorem 2.2 is easy.

Suppose $E\subset L_\varphi^{**,r}$, for suitable $t>0$, we introduce the following best degree of approximation

$$\rho_{r,\varphi}(f,E)=\inf_{g\in E}\left\|f-g\right\|_{r,\langle\varphi\rangle,t}. \qquad (2.4)$$

Theorem 2.3. *Suppose that $r=0,1,\ldots$, $\upsilon=0,1,\ldots,r$, $t>0$ and $f\in L_\varphi^{**,r}$. Then*

$$c_1t^\upsilon\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right)\le K_{r,\varphi}^\upsilon(f,t)\le c_2t^\upsilon\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right) \qquad (2.5)$$

where the constants c_1 and c_2 are depending only on r and φ.

Proof of Theorem 2.3. First, let us prove the lower estimate. We will follow the method given in (Garidi, 1991). For $0\le\upsilon\le r$, $f\in L_\varphi^{**,r}$ and any $g\in L_\varphi^{**,r-\upsilon}$, from the property (3) of $\omega_{r,\langle\varphi\rangle}(f,t)$, we obtain

$$\begin{aligned}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right)&\le\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)}-g,t\right)+\omega_{r-\upsilon,\langle\varphi\rangle}(g,t)\\ &\le 2^{r-\upsilon}\omega_{0,\langle\varphi\rangle}\left(f^{(\upsilon)}-g,t\right)+t^{r-\upsilon}\omega_{0,\langle\varphi\rangle}\left(g^{(r-\upsilon)},t\right)\\ &\le c\left(\left\|f^{(\upsilon)}-g\right\|_{\langle\varphi\rangle}+t^{r-\upsilon}\left\|g^{(r-\upsilon)}\right\|_{\langle\varphi\rangle}\right).\end{aligned}$$

Since g is arbitrary we have

$$ct^\upsilon\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right)\le t^\upsilon K_{r-\upsilon,\langle\varphi\rangle}^0\left(f^{(\upsilon)},t\right). \qquad (2.6)$$

and for $f\in L_\varphi^{**,r}$, $i=0,1,\ldots,\upsilon-1$ there holds

$$\begin{aligned}K_{r-t,\varphi}^{\upsilon-t}\left(f^{(i)},t\right)&=\inf\left\{\left\|f^{(i)}-g\right\|_{\upsilon-t,\langle\varphi\rangle}+t^{r-t}\left\|g^{(r-t)}\right\|_{\upsilon-t,\langle\varphi\rangle}:g\in L_\varphi^{**,(r+\upsilon-2t)}\right\}\\ &\ge\inf\left\{t.\left\|f^{(i+1)}-g^{(1)}\right\|_{\upsilon-t-1,\langle\varphi\rangle}+t^{r-t}\left\|g^{(r-t)}\right\|_{\upsilon-t-1,\langle\varphi\rangle}:g\in L_\varphi^{**,(r+\upsilon-2t)}\right\}\\ &\ge t.\inf\left\{\left\|f^{(i+1)}-h\right\|_{\upsilon-t-1,\langle\varphi\rangle}+t^{r-t-1}\left\|h^{(r-t-1)}\right\|_{\upsilon-t-1,\langle\varphi\rangle}:h\in L_\varphi^{**,(r+\upsilon-2t-2)}\right\}\\ &=t.K_{r-t-1,\varphi}^{\upsilon-t-1}\left(f^{(i+1)},t\right).\end{aligned} \qquad (2.7)$$

From this recurrence formula, we obtain

$$K_{r,\varphi}^\upsilon(f,t)\ge t^\upsilon.K_{r-\upsilon,\varphi}^0\left(f^{(\upsilon)},t\right). \qquad (2.8)$$

Thus, from (2.6) and (2.8), it follows the lower estimate.

$$ct^\upsilon\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right)\le K_{r,\varphi}^\upsilon(f,t).$$

Now, let us prove the upper estimate. For $r=0$, in view of $K_{0,\varphi}^0(f,t)=\|f\|_{\langle\varphi\rangle}=\omega_{0,\langle\varphi\rangle}(f,t)$, the conclusion of Theorem 2.3 is obvious. In the following, we always assume that $r\ge 1$ and $f\in L_\varphi^{**,r}$. First, we suppose $0\le t\le\frac{b-a}{4r^2}$, and we define

$$I_0=\left[a,a+\frac{3(b-a)}{4}\right],\ I_1=\left[a+\frac{b-a}{4},b\right],\ I_2=I_0\cap I_1=\left[a+\frac{b-a}{4},a+\frac{3(b-a)}{4}\right],$$

$$g_0(x)=f(x)+t^{-r}\int_0^t\ldots\int_0^t(-1)^{r+1}\Delta^r_{u_1+\ldots+u_r}(f,x)\,du_1\ldots du_r,\ x\in I_0,$$

$$g_1(x)=f(x)+t^{-r}\int_0^t\ldots\int_0^t(-1)^{r+1}\Delta^r_{-(u_1+\ldots+u_r)}(f,x)\,du_1\ldots du_r,\ x\in I_1.$$

Suppose $\int_{I_0}\Psi_1(v(x))dx\le 1$. Then for $i=0,1,\ldots,r$, we have

$$\left|\int_{I_0}\left(f^{(i)}(x)-g_0^{(i)}(x)\right)v(x)dx\right|=\left|\int_{I_0}\left(t^{-r}\int_0^t\ldots\int_0^t\Delta^r_{u_1+\ldots+u_r}\left(f^{(i)},x\right)du_1\ldots du_r\right)v(x)dx\right|$$

$$=\left|t^{-r}\int_0^t\ldots\int_0^t\left(\int_{I_0}\Delta^r_{u_1+\ldots+u_r}\left(f^{(i)},x\right)v(x)dx\right)du_1\ldots du_r\right|$$

$$\le t^{-r}\int_0^t\ldots\int_0^t\omega_{r,\varphi}\left(f^{(i)},rt\right)du_1\ldots du_r$$

$$=\omega_{r,\langle\varphi\rangle}\left(f^{(i)},rt\right)\le r^r\omega_{r,\langle\varphi\rangle}\left(f^{(i)},t\right).$$

Hence

$$\left\|f^{(i)}-g_0^{(i)}\right\|_{\langle\varphi\rangle(I_0)}\le r^r\omega_{r,\langle\varphi\rangle}\left(f^{(i)},t\right)\le c.t^{\upsilon-i}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right).\tag{2.9}$$

By similar arguments, we get

$$\left\|f^{(i)}-g_1^{(i)}\right\|_{\langle\varphi\rangle(I_1)}\le c.t^{\upsilon-i}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right),\tag{2.10}$$

$$t^r\left\|g_0^{(r+i)}\right\|_{\langle\varphi\rangle(I_0)}=\left\|\sum_{j=1}^r(-1)^{r+j}\binom{r}{j}j^{-r}\Delta^r_{ji}\left(f^{(i)},.\right)\right\|_{\langle\varphi\rangle(I_0)}$$

$$\le\sum_{j=1}^r\binom{r}{j}j^{-r}\omega_{r,\langle\varphi\rangle}\left(f^{(i)},jt\right)\le ct^{\upsilon-i}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right)\tag{2.11}$$

and

$$t^r\left\|g_1^{(r+i)}\right\|_{\langle\varphi\rangle(I_1)}\le ct^{\upsilon-i}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right).\tag{2.12}$$

Let us take a function $\varphi(x)$ with $\varphi(x)=0$ for $x\in\left[a,a+\frac{b-a}{4}\right]$, $\varphi(x)=1$ for $x\in\left[a+\frac{3(b-a)}{4},b\right]$ and $\left|\varphi^{(i)}(x)\right|\le c$ for $i=0,1,\ldots,2r$, $x\in[a,b]$. We extend $g_0\in L_\varphi^{**,r+\upsilon}(I_0)$ and $g_1\in L_\varphi^{**,r+\upsilon}(I_1)$ according to $g_0,g_1\in L_\varphi^{**,r+\upsilon}$ (This is possible by a method of Ramazanov (1984)). Let

$$g(x):=(1-\varphi(x))g_0(x)+\varphi(x)g_1(x).\tag{2.13}$$

Then, for $i=0,1,\ldots,r+\upsilon$, we have

$$g^{(i)}(x)=(1-\varphi(x))g_0^{(i)}(x)+\varphi(x)g_1^{(i)}(x)+\sum_{j=0}^{i-1}\varphi^{(i-j)}(x)\left(g_1^{(j)}(x)-g_0^{(j)}(x)\right)\tag{2.14}$$

hence, for $i=0,1,\ldots,r+\upsilon$, from (2.9), (2.10) and (2.14), we have

$$\left\|f^{(i)}-g^{(i)}\right\|_{\langle\varphi\rangle(I-I_0)}=\left\|f^{(i)}-g_1^{(i)}\right\|_{\langle\varphi\rangle(I-I_0)}\le ct^{\upsilon-i}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right),\tag{2.15}$$

$$\left\|f^{(i)}-g^{(i)}\right\|_{\langle\varphi\rangle(I-I_1)}=\left\|f^{(i)}-g_0^{(i)}\right\|_{\langle\varphi\rangle(I-I_1)}\le ct^{\upsilon-i}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right),\tag{2.16}$$

$$\left\|f^{(i)}-g^{(i)}\right\|_{\langle\varphi\rangle(I_2)} \le \left\|f^{(i)}-g_0^{(i)}\right\|_{\langle\varphi\rangle(I_0)} + \left\|f^{(i)}-g_1^{(i)}\right\|_{\langle\varphi\rangle(I_1)}$$

$$+c\sum_{j=0}^{i}\left\|g_1^{(j)}-g_0^{(j)}\right\|_{\langle\varphi\rangle(I_2)}$$

$$\le ct^{\upsilon-i}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right)+c\sum_{j=0}^{i}\left(\left\|f^{(j)}-g_0^{(j)}\right\|_{\langle\varphi\rangle(I_0)}+\left\|f^{(j)}-g_1^{(j)}\right\|_{\langle\varphi\rangle(I_1)}\right)$$

$$\le ct^{\upsilon-i}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right). \qquad (2.17)$$

For $i=0,1,...,r$ from (2.9) to (2.14) and Lemma 1a in Ramazanov (1984), we deduce

$$\left\|g^{(r+i)}\right\|_{\langle\varphi\rangle(I)} \le c\left(\left\|g_0^{(r+i)}\right\|_{\langle\varphi\rangle(I_0)}+\left\|g_1^{(r+i)}\right\|_{\langle\varphi\rangle(I_1)}+\right.$$

$$\left.+\sum_{j=0}^{r+i}\varphi^{(i-j)}(x)\left\|g_1^{(j)}(x)-g_0^{(j)}(x)\right\|_{\langle\varphi\rangle(I_2)}\right)$$

$$\le c\left(\left\|g_0^{(r+i)}\right\|_{\langle\varphi\rangle(I_0)}+\left\|g_1^{(r+i)}\right\|_{\langle\varphi\rangle(I_1)}+\left\|g_1-g_0\right\|_{\langle\varphi\rangle(I_2)}\right)$$

$$\le c\left(\left\|g_0^{(r+i)}\right\|_{\langle\varphi\rangle(I_0)}+\left\|g_1^{(r+i)}\right\|_{\langle\varphi\rangle(I_1)}+\left\|f-g_0\right\|_{\langle\varphi\rangle(I_0)}+\left\|f-g_1\right\|_{\langle\varphi\rangle(I_1)}\right)$$

$$\le ct^{\upsilon-i-r}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right). \qquad (2.18)$$

Hence using

$$\|.\|_{\langle\varphi\rangle(I)}=\|.\|_{\langle\varphi\rangle(I_2)}+\|.\|_{\langle\varphi\rangle(I-I_0)}+\|.\|_{\langle\varphi\rangle(I-I_1)}$$

from (2.15) to (2.18), for $0<t<\frac{b-a}{4r^2}$ we have

$$K_{r,\varphi}^{\upsilon}(f,t)\le\|f-g\|_{\upsilon,\langle\varphi\rangle}+t^r\left\|g^{(r)}\right\|_{\upsilon,\langle\varphi\rangle}\le ct^{\upsilon}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right). \qquad (2.19)$$

On the other hand, for any $s\ge 1$ we have

$$K_{r,\varphi}^{\upsilon}(f,st)\le s^{r+\upsilon}K_{r,\varphi}^{\upsilon}(f,t).$$

Hence, from (2.19) and monotonicity of $t^{\upsilon}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right)$, we get

$$K_{r,\varphi}^{\upsilon}(f,st)\le s^{r+\upsilon}ct^{\upsilon}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right)\le\left(s^{r+\upsilon}c\right)(st)^{\upsilon}\,\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},st\right).$$

Thus, for $0<t\le b-a$, we have

$$K_{r,\varphi}^{\upsilon}(f,st)\le ct^{\upsilon}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right) \qquad (2.20)$$

where c is a constant depending on r only. Obviously, $P_{r-1}\subset L_{\varphi}^{**,r+\upsilon}$ and for any $p(x)\in P_{r-1}$ and $i=0,1,...,\upsilon$, $p^{(r+i)}(x)=0$. Hence, for $r\ge1$, $K_{r+\varphi}^{\upsilon}(f,t)\le\rho_{\upsilon,\varphi}(f,P_{r-1})$. Thus, by Lemma 2.4 given in below, we can easily prove that Theorem 2.3 is also true for $t\ge b-a$. Now the proof of Theorem 2.3 is completed.

Lemma 2.4. *Suppose that $r=1,2,...$ and $\upsilon=0,1,...,r$, then for any $f\in L_{\varphi}^{**,r}$ and $t\ge b-a$, we have*

$$\rho_{r,\varphi}(f,P_{r-1})\le ct^{\upsilon}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right)$$

where P_{r-1} is the set of all algebraic polynomials of degree $r-1$ and c depends on r and φ only.

Proof of Lemma 2.4. From the first part of the proof of Theorem 2.3, it follows for $f\in L_{\varphi}^{**,r}$ that there exists $g\in L_{\varphi}^{**,r+\upsilon}$ such that

$$\left\|f-g\right\|_{\upsilon,\langle\varphi\rangle,(b-a)}+(b-a)^{r}\left\|g^{(r)}\right\|_{\upsilon,\langle\varphi\rangle,(b-a)}\le c(b-a)^{r}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},b-a\right). \tag{2.21}$$

Like in Garidi (1991) we set $p(x)=\sum_{i=0}^{r-1}\frac{g^{(i)}(a)}{i!}(x-a)^{i}\in P_{r-1}$. Then for $i=1,...,r$ and $x\in[a,b]$, we have

$$\left|g^{(i-1)}(x)-p^{(i-1)}(x)\right|=\left|\int_a^x\left(g^{(i)}(t)-p^{(i)}(t)\right)dt\right|$$
$$\le\int_a^x\left|\left(g^{(i)}(t)-p^{(i)}(t)\right)\right|dt\le\left\|g^{(i)}-p^{(i)}\right\|_{\langle\varphi\rangle}\left\|1\right\|_{\Psi_1}.$$

Thus, for $i=1,...,r$, it is easy to see that

$$\left\|g^{(i-1)}-p^{(i-1)}\right\|_{\langle\varphi\rangle}=\sup_{\rho(v,\Psi_1)\le1}\left|\int_a^b\left(g^{(i-1)}(x)-p^{(i-1)}(x)\right)v(x)dx\right|$$
$$\le\left\|g^{(i)}-p^{(i)}\right\|_{\langle\varphi\rangle}\left\|1\right\|_{\Psi_1}\sup_{\rho(v,\Psi_1)\le1}\int_a^b\left|v(x)\right|dx$$
$$=\left\|1\right\|_{\langle\varphi\rangle}\left\|1\right\|_{\Psi_1}\left\|g^{(i)}-p^{(i)}\right\|_{\langle\varphi\rangle}\le c(b-a)\left\|g^{(i)}-p^{(i)}\right\|_{\langle\varphi\rangle}.$$

Hence, for $t\ge b-a$, we can obtain

$$\left\|f-p\right\|_{\upsilon,\langle\varphi\rangle,t}\le\left\|f-g\right\|_{\upsilon,\langle\varphi\rangle,t}+\left\|g-p\right\|_{\upsilon,\langle\varphi\rangle,t}$$
$$\le\left\|f-g\right\|_{\upsilon,\langle\varphi\rangle,t}+\left[c(b-a)\right]^{r}\left\|g^{(r)}\right\|_{\langle\varphi\rangle}\sum_{i=0}^{\upsilon}\left(\frac{t}{c(b-a)}\right)^{i}$$
$$\le c\left\{\left\|f-g\right\|_{\upsilon,\langle\varphi\rangle,t}+(b-a)^{r}\left\|g^{(r)}\right\|_{\langle\varphi\rangle}\sum_{i=0}^{\upsilon}\left(\frac{t}{(b-a)}\right)^{i}\right\}$$
$$\le c\left(\frac{t}{(b-a)}\right)^{\upsilon}\left\{\left\|f-g\right\|_{\upsilon,\langle\varphi\rangle,(b-a)}+(b-a)^{r}\left\|g^{(r)}\right\|_{\langle\varphi\rangle}(\upsilon+1)\right\}$$
$$\le c\left(\frac{t}{(b-a)}\right)^{\upsilon}\left\{\left\|f-g\right\|_{\upsilon,\langle\varphi\rangle,(b-a)}+(b-a)^{r}\left\|g^{(r)}\right\|_{\upsilon,\langle\varphi\rangle,(b-a)}\right\}. \tag{2.22}$$

In view of (2.21), (2.22) and the following inequality

$$(b-a)^{\upsilon}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},b-a\right)\le t^{r}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right)$$

for $t\ge b-a$, we complete the proof of Lemma 2.4.

Corollary 2.5. *Suppose that L is a bounded linear operator from L_{φ}^{**} to L_{φ}^{**} and $\left\|L\right\|_{\upsilon,\langle\varphi\rangle}\le c_0$. If for any $g\in L_{\varphi}^{**,r+\upsilon}$ the inequality*

$$\left\|g-L(g)\right\|_{\upsilon,\langle\varphi\rangle}\le c_0t^{r}\left\|g^{(r)}\right\|_{\upsilon,\langle\varphi\rangle}$$

holds, then for any $f\in L_{\varphi}^{**,r}$, we have

$$\left\|f-L(f)\right\|_{\upsilon,\langle\varphi\rangle}\le ct^{r}\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},t\right),$$

where $\left\|L\right\|_{\upsilon,\langle\varphi\rangle}=\sup_{\left\|f\right\|_{\upsilon,\langle\varphi\rangle}\le1}\left\|L(f)\right\|_{\upsilon,\langle\varphi\rangle}$ and c depends on r and φ only.

3. Proof of the main results

For the proof of the main results we will need the following Lemmas.

Lemma 3.1. *Suppose that $I=[a,b]$ and $J=[c,d]$ are closed intervals such that $I\subset J$, $r=0,1,...,$ and $f(x)$ is a function defined on I. We extend $f(x)$ from I to J by the following formula*

$$f_0(x)=\begin{cases} f(x) & , x\in[a,b] \\ \sum_{i=0}^{2r}\alpha_i f\left(a+2^{-i}\frac{b-a}{a-c}(a-x)\right) & , x\in[c,a) \\ \sum_{i=0}^{2r}\beta_i f\left(b-2^{-i}\frac{b-a}{d-b}(x-b)\right) & , x\in(b,d] \end{cases}$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are real numbers which satisfy the following conditions

$$\sum_{i=0}^{2r}\alpha_i\left(-2^{-i}\frac{b-a}{a-c}\right)^j=1,\ \sum_{i=0}^{2r}\beta_i\left(-2^{-i}\frac{b-a}{d-b}\right)^j=1,\ j=0,1,...,2r.$$

Then for $i,j=0,1,...,r,t>0$ and any $f\in L_\varphi^{**,i}$, we have

$$\omega_{j,\langle\varphi\rangle(J)}\left(f_0^{(i)},t\right)\le c\omega_{j,\langle\varphi\rangle(I)}\left(f^{(i)},t\right),$$

where c depends on r, I and J.

The proof of Lemma 3.1 can be done by using the method of that Lemma in Ramazanov (1984) and Theorem 2 (Ramazanov, 1984). Let P_n be the set of all algebraic polynomials of degree n.

We take $\lambda_n(t)\in P_n$ such that

$$\int_{\frac{-1}{4r}}^{\frac{1}{4r}}\lambda_n(t)\,dt=1. \tag{3.1}$$

Let

$$K_n(t):=\lambda_n\left(\frac{t}{4}\right)/4, \tag{3.2}$$

$$\varphi_n(t):=\sum_{k=1}^{r}(-1)^{k+1}\binom{r}{k}K_n\left(\frac{t}{k}\right)/k. \tag{3.3}$$

If $f(x)$ is a function defined on $[-1,1]$, we extend $f(x)$ to $[-2,2]$ by using the method of Lemma 3.1 and denote the extended function of $f(x)$ by $f_0(x)$.

Let

$$\Phi_n(f,x)=\int_{-2}^{2}f_0(t)\varphi_n(t-x)\,dt\in P_n, \tag{3.4}$$

$$F(x)=\int_0^x f(t)\,dt.$$

Assume that $Q_r(x)$ is a r-th interpolation polynomial of $F(x)$ with interpolation nodes $-1+\frac{2i}{r}$, $i=1,2,...,r$.

Set $q(f,x):=Q_r'(x)$. We define an important polynomial as following.

$$L_n(f,x)=\Phi_n(f-q(f),x)+q(f,x)\in P_n,\ n\ge 1. \tag{3.5}$$

Lemma 3.2. (Yiqun, 1984) *Suppose that $r,n=1,2,...$ and $i=0,1,...,4r-2$. If the K_n is defined by (3.2) then satisfies*

$$\int_{-4}^{4}|t|^i K_n(t)\,dt\le cn^{-i}, \tag{3.6}$$

$$\left|K_n^{(i)}(t)\right|\le cn^{1-4r+i},\ \ t\in[-3,3]/\left[-\frac{1}{r},\frac{1}{r}\right] \tag{3.7}$$

where c depends on r only.

Lemma 3.3. *Suppose that $r=1,2,...$ and $\upsilon=0,1,...,r$. Then for the linear operator Φ_n which defined by (3.4) and any $f\in L_\varphi^{**,r}$ which has at least υ zeros on $[-1,1]$, we have*

$$\left\|f^{(\upsilon)}(.)-\Phi_n^{(\upsilon)}(f,.)\right\|_{\langle\varphi\rangle}\le c\left\{\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},\frac{1}{n}\right)+n^{1-4r+\upsilon}\left\|f^{(\upsilon)}\right\|_{\langle\varphi\rangle}\right\}$$

where c depends on r only.

Proof of Lemma 3.3. Following the arguments in Garidi (1991) let

$$E_{i,x}=\left[\frac{-2-x}{i},\frac{2-x}{i}\right]/\left(-\frac{1}{r},\frac{1}{r}\right),\ R_{i,x}=\int_{E_{i,x}}f_0(x+it)K_n(t)\,dt$$

then, we have

$$\Phi_n(f,x)=\int_{-\frac{1}{r}}^{\frac{1}{r}}\left\{f_0(x)+(-1)^{r+1}\Delta_t^r(f_0,x)\right\}K_n(t)\,dt+\sum_{i=1}^{r}(-1)^{i+1}\binom{r}{i}R_i(x)$$

$$=A_n(f,x)+B_n(f,x).$$

Therefore

$$\left\|f^{(\upsilon)}(.)-\Phi_n^{(\upsilon)}(f,.)\right\|_{\langle\varphi\rangle}\le\left\|f^{(\upsilon)}(.)-A_n^{(\upsilon)}(f,.)\right\|_{\langle\varphi\rangle}+\left\|B_n^{(\upsilon)}(f,.)\right\|_{\langle\varphi\rangle}.$$

Assume that $g\in L_\varphi^{**,r+\upsilon}$ and g_0 is the extended function of g from [–1,1] to [–2,2]. Then, it follows that from (3.1),

$$g^{(i)}(x)-A_n^{(i)}(g,x)=(-1)^r\int_{-\frac{1}{r}}^{\frac{1}{r}}\Delta_t^r\left(g_0^{(i)},x\right)K_n(t)\,dt,\;\; i=0,1,...,\nu.$$

For $i=0,1,...,\upsilon$, from Lemma 3.1 and (3.6), we have

$$\left\|g^{(i)}(.)-A_n^{(i)}(g,.)\right\|_{\langle\varphi\rangle}=\sup_{\rho(v,\Psi_1)\le1}\left|\int_{-1}^{1}\left(\int_{-\frac{1}{r}}^{\frac{1}{r}}\Delta_t^r\left(g_0^{(i)},x\right)K_n(t)\,dt\right)v(x)\,dx\right|$$

$$\le\int_{-\frac{1}{r}}^{\frac{1}{r}}\left(\sup_{\rho(v,\Psi_1)\le1}\left|\int_{-1}^{1}\Delta_t^r\left(g_0^{(i)},x\right)v(x)\,dx\right|\right)K_n(t)\,dt$$

$$\le\int_{-\frac{1}{r}}^{\frac{1}{r}}\omega_{r,\langle\varphi\rangle}\left(g_0^{(i)},t\right)K_n(t)\,dt\le\int_{-\frac{1}{r}}^{\frac{1}{r}}|t|^r\left\|g_0^{(i+r)}\right\|_{\langle\varphi\rangle[-2,2]}K_n(t)\,dt$$

$$\le c\left\|g^{(i+r)}\right\|_{\langle\varphi\rangle[-1,1]}\int_{-\frac{1}{r}}^{\frac{1}{r}}|t|^r K_n(t)\,dt\le cn^{-r}\left\|g^{(i+r)}\right\|_{\langle\varphi\rangle[-1,1]}.$$

Hence, for $t=\frac{1}{n}$ and any $g\in L_\varphi^{**,r+\upsilon}$, we have

$$\left\|g(.)-A_n(g,.)\right\|_{\upsilon,\langle\varphi\rangle,\frac{1}{n}}\le c\left(\frac{1}{n}\right)^r\left\|g^{(r)}\right\|_{\upsilon,\langle\varphi\rangle,\frac{1}{n}}. \tag{3.8}$$

On the other hand, for $f\in L_\varphi^{**,r+\upsilon}[-1,1]$ and $i=0,1,...,\upsilon$, we easily get

$$\left\|A_n^{(i)}(f,.)\right\|_{\langle\varphi\rangle}=\left\|f^{(i)}(x)+\sum_{j=0}^{r}\int_{-\frac{1}{r}}^{\frac{1}{r}}(-1)^{r-j}\binom{r}{j}f_0^{(i)}(x+jt)K_n(t)\,dt\right\|_{\langle\varphi\rangle}$$

$$\le\left\|f^{(i)}(x)\right\|_{\langle\varphi\rangle}+\sup_{\rho(v,\Psi_1)\le1}\left|\int_{-1}^{1}\left\{\sum_{j=0}^{r}\int_{-\frac{1}{r}}^{\frac{1}{r}}(-1)^{r-j}\binom{r}{j}f_0^{(i)}(x+jt)K_n(t)\,dt\right\}v(x)\,dx\right|$$

$$\le\left\|f^{(i)}(x)\right\|_{\langle\varphi\rangle}+\sum_{j=0}^{r}\binom{r}{j}\int_{-\frac{1}{r}}^{\frac{1}{r}}\left\{\sup_{\rho(v,\Psi_1)\le1}\left|\int_{-1+jt}^{1+jt}f_0^{(i)}(y)v(y-jt)\,dy\right|\right\}K_n(t)\,dt$$

$$\leq \left\| f^{(i)}(x) \right\|_{\langle\varphi\rangle} + \sum_{j=0}^{r} \binom{r}{j} \int_{-\frac{1}{r}}^{\frac{1}{r}} \left\{ \sup_{\rho(v,\Psi_1)\leq 1} \left\| f_0^{(i)} \right\|_{\langle\varphi\rangle[-1+jt,1+jt]} \left\| v \right\|_{\langle\Psi_1\rangle[-1,1]} \right\} K_n(t)\,dt$$

$$\leq \left\| f^{(i)}(x) \right\|_{\langle\varphi\rangle} + c\left\| f_0^{(i)} \right\|_{\langle\varphi\rangle[-2,2]} \int_{-\frac{1}{r}}^{\frac{1}{r}} K_n(t)\,dt \leq c\left\| f^{(i)} \right\|_{\langle\varphi\rangle[-1,1]}.$$

Thus, we obtain

$$\left\| A_n(f,.) \right\|_{\upsilon,\langle\varphi\rangle} \leq c\left\| f \right\|_{\upsilon,\langle\varphi\rangle},\ f \in L_{\varphi}^{**,r}[-1,1]$$

which implies

$$\left\| A_n \right\|_{\upsilon,\langle\varphi\rangle} \leq c, \quad (n=1,2,...). \tag{3.9}$$

(3.8), (3.9) and from the corollary of Theorem 2.3, for $f \in L_{\varphi}^{**,r}[-1,1]$

$$\left\| f(.) - A_n(f,.) \right\|_{\upsilon,\langle\varphi\rangle,\frac{1}{n}} \leq c\left(\frac{1}{n}\right)^{\upsilon} \omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},\frac{1}{n}\right).$$

Therefore, it follows that

$$\left\| f^{(\upsilon)}(.) - A_n^{(\upsilon)}(f,.) \right\|_{\langle\varphi\rangle} \leq c\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)},\frac{1}{n}\right). \tag{3.10}$$

In addition, in view of

$$B_n^{(\upsilon)}(f,x) = \sum_{i=0}^{r} (-1)^{i+1} \binom{r}{i} R_i^{(\upsilon)}(x)$$

and

$$R_i^{(\upsilon)}(x) = \int_{E_{i,x}} f_0^{(\upsilon)}(x+it) K_n(t)\,dt +$$

$$+\sum_{j=0}^{\upsilon-1} \left\{ \left(1-\frac{1}{i}\right)^{j+1} \left[f_0^{(\upsilon-1-j)}(2) K_n^{(j)}\left(\frac{2-x}{i}\right) - f_0^{(\upsilon-1-j)}(-2) K_n^{(j)}\left(\frac{-2-x}{i}\right) \right] \right\}$$

from (3.7), we can obtain

$$\left\| \int_{E_{i,x}} f_0^{(\upsilon)}(x+it) K_n(t)\,dt \right\|_{\langle\varphi\rangle} = \sup_{\rho(v,\Psi_1)\leq 1} \left| \int_{-1}^{1} \left\{ \int_{E_{i,x}} f_0^{(\upsilon)}(x+it) K_n(t)\,dt \right\} v(x)\,dx \right|$$

$$\leq \int_{E_{i,x}} \left\{ \sup_{\rho(v,\Psi_1)\leq 1} \left| \int_{-1+it}^{1+it} f_0^{(\upsilon)}(y) v(y-it)\,dy \right| \right\} K_n(t)\,dt$$

$$\leq c\left\| f_0^{(\upsilon)} \right\|_{\langle\varphi\rangle[-2,2]} \int_{E_{i,x}} K_n(t)\,dt \leq cn^{1-4r} \left\| f^{(\upsilon)} \right\|_{\langle\varphi\rangle[-1,1]}$$

$$\leq cn^{1-4r+\upsilon} \left\| f^{(\upsilon)} \right\|_{\langle\varphi\rangle}. \tag{3.11}$$

Thus, by the assumptions, we have that $f^{(i)}$ have zeros on $[-1,1]$ for $i=1,2,...,\upsilon-1$.

Suppose that the zeros of $f^{(i)}$ are c_i for $i=1,2,...,\upsilon-1$. Then, for $x\in[-2,2]$, we have

$$\left|f_0^{(i)}(x)\right|=\left|\int_{c_i}^{x} f_0^{(i+1)}(t)\,dt\right|\le\int_{-2}^{2}\left|f_0^{(i+1)}(x)\right|dx$$

$$\le\left\|f_0^{(i+1)}\right\|_{\langle\varphi\rangle[-2,2]}\|1\|_{\langle\Psi_1\rangle}\le c\left\|f^{(i+1)}\right\|_{\langle\varphi\rangle[-1,1]}.$$

Hence, it follows that for $i=0,1,...,\upsilon-1$

$$\left\|f^{(i)}\right\|_{\langle\varphi\rangle[-1,1]}\le c\left\|f^{(i+1)}\right\|_{\langle\varphi\rangle[-1,1]}.$$

Thus for $x\in[-2,2]$, we have

$$\left|f_0^{(i)}(x)\right|\le c\left\|f^{(i+1)}\right\|_{\langle\varphi\rangle}\le...\le c\left\|f^{(\upsilon)}\right\|_{\langle\varphi\rangle},\ i=0,1,...,\upsilon-1.$$

Therefore, for $j=0,1,...,\upsilon-1$

$$\left|f_0^{(\upsilon-1-j)}(2)\right|\le c\left\|f^{(\upsilon)}\right\|_{\langle\varphi\rangle},\left|f_0^{(\upsilon-1-j)}(-2)\right|\le c\left\|f^{(\upsilon)}\right\|_{\langle\varphi\rangle}.$$

By using (3.7) again, we get

$$\left|\sum_{j=0}^{\upsilon-1}\left\{\left(-\frac{1}{i}\right)^{j+1}\left[f_0^{(\upsilon-1-j)}(2)K_n^{(j)}\left(\frac{2-x}{i}\right)-f_0^{(\upsilon-1-j)}(-2)K_n^{(j)}\left(\frac{-2-x}{i}\right)\right]\right\}\right|$$

$$\le cn^{1-4r+\upsilon}\left\|f^{(\upsilon)}\right\|_{\langle\varphi\rangle}.$$

Hence, we have

$$\left\|\sum_{j=0}^{\upsilon-1}\left\{\left(-\frac{1}{i}\right)^{j+1}\left[f_0^{(\upsilon-1-j)}(2)K_n^{(j)}\left(\frac{2-x}{i}\right)-f_0^{(\upsilon-1-j)}(-2)K_n^{(j)}\left(\frac{-2-x}{i}\right)\right]\right\}\right\|_{\langle\varphi\rangle}$$

$$\le cn^{1-4r+\upsilon}\left\|f^{(\upsilon)}\right\|_{\langle\varphi\rangle}. \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\left\|R_i^{(\upsilon)}(.)\right\|_{\langle\varphi\rangle}\le cn^{1-4r+\upsilon}\left\|f^{(\upsilon)}\right\|_{\langle\varphi\rangle},\quad i=1,...,r.$$

Therefore,

$$\left\|B_n^{(\upsilon)}(f,.)\right\|_{\langle\varphi\rangle}\le cn^{1-4r+\upsilon}\left\|f^{(\upsilon)}\right\|_{\langle\varphi\rangle}. \tag{3.13}$$

From (3.10) and (3.13), we can complete the proof of Lemma 3.3.

Lemma 3.4. *Suppose that* $r=1,2,...,\ \upsilon=0,1,...,r$ *and* $q(f,x)$ *is the same as above.*

Let $\left\|q^{(\upsilon)}\right\|=\sup_{\|f\|_{\langle\varphi\rangle}\le1}\left\|q^{(\upsilon)}(f,.)\right\|_{\langle\varphi\rangle}$. *Then, for any* $f\in L_\varphi^{**,r}$, *we have*

$$\left\|f^{(\upsilon)}(.)-q^{(\upsilon)}(f,.)\right\|_{\langle\varphi\rangle}\le c\left\{1+\left\|q^{(\upsilon)}\right\|(b-a)^{\upsilon}\right\}\omega_{r-\upsilon,\varphi}\left(f^{(\upsilon)},b-a\right)$$

where c depends on r only.

Proof of Lemma 3.4. Assume that $Q(x)\in P_{r-1}$. Then, by using Lemma 2.4, we obtain (Garidi, 1991)

$$\left\|f^{(\upsilon)}(.)-q^{(\upsilon)}(f,.)\right\|_{\langle\varphi\rangle}\le\left\|f^{(\upsilon)}-Q^{(\upsilon)}\right\|_{\langle\varphi\rangle}+\left\|q^{(\upsilon)}(f-Q,.)\right\|_{\langle\varphi\rangle}$$

$$\le \left\{ (b-a)^{-\upsilon} + \left\| q^{(\upsilon)} \right\| \right\} \left\| f^{(\upsilon)} - Q^{(\upsilon)} \right\|_{\upsilon,\langle\varphi\rangle(b-a)}$$

$$\le c\left\{ 1 + \left\| q^{(\upsilon)} \right\| (b-a)^{\upsilon} \right\} \omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)}, b-a \right).$$

Proof of Theorem 1.1. Suppose that $\upsilon = 0,1,...,r$ for $x \in [-1,1]$. By the Markov's inequality, we have

$$\left| q^{(\upsilon)}(f,x) \right| \le c \max_{x\in[-1,1]} \left| q(f,x) \right| = c \max_{x\in[-1,1]} \left| Q_r'(x) \right| \le c \max_{x\in[-1,1]} \left| Q_r(x) \right|. \tag{3.14}$$

By using Lemma 4 in Yiqun (1984), we obtain

$$c \max_{x\in[-1,1]} \left| Q_r(x) \right| \le c \max_{x\in[-1,1]} \left| F(x) \right| = c \max_{x\in[-1,1]} \left| \int_0^x f(t)\,dt \right|$$

$$\le c \int_{-1}^{1} \left| f(x) \right| dx \le c \left\| f \right\|_{\langle\varphi\rangle} \left\| 1 \right\|_{\langle\Psi_1\rangle} \le c \left\| f \right\|_{\langle\varphi\rangle}. \tag{3.15}$$

From (3.14) and (3.15), it follows that

$$\left\| q^{(\upsilon)}(f,.) \right\|_{\langle\varphi\rangle} \le c \left\| f \right\|_{\langle\varphi\rangle}, \quad \left\| q^{(\upsilon)} \right\|_{\langle\varphi\rangle} \le c.$$

By Lemma 3.3, Lemma 3.4 and notice that $q(f,x)$ is a polynomial of degree $r-1$, (see for example Garidi, 1991) we have

$$\left\| f^{(\upsilon)}(.) - L_n^{(\upsilon)}(f,.) \right\|_{\langle\varphi\rangle} = \left\| f^{(\upsilon)}(.) - q^{(\upsilon)}(f,.) - \Phi_n^{(\upsilon)}\left(f - q(f),. \right) \right\|_{\langle\varphi\rangle}$$

$$\le c\left\{ \omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)}, \frac{1}{n} \right) + n^{1-4r+\upsilon} \left\| f^{(\upsilon)}(.) - q^{(\upsilon)}(f,.) \right\|_{\langle\varphi\rangle} \right\}$$

$$\le c\left\{ \omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)}, \frac{1}{n} \right) + n^{1-4r+\upsilon} \omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)}, 2 \right) \right\}$$

$$\le c\left\{ \omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)}, \frac{1}{n} \right) + n^{1-4r+\upsilon} (2n)^{r-\upsilon} \omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)}, \frac{1}{n} \right) \right\} \le c\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)}, \frac{1}{n} \right).$$

Proof of Theorem 1.2. From chapter 4 in Lorentz (1966), it follows that for $r = 1,2,...$, there exists $\left\{ \tilde{K}_n(t) \right\} \subset T_n$, such that T_n is a trigonometric polynomial of degree *n,* we have

$$\int_{-\pi}^{\pi} \tilde{K}_n(t)\,dt = 1, \tag{3.16}$$

$$\int_{-\pi}^{\pi} |t|^i \left| \tilde{K}_n(t) \right| dt \le cn^{-i}, \quad i = 0,1,...,r. \tag{3.17}$$

Let (Garidi, 1991)

$$L_n(f,x) := \int_{-\pi}^{\pi} \left\{ f(x) + (-1)^{r+1} \Delta_t^r(f,x) \right\} \tilde{K}_n(t)\,dt \in T_n. \tag{3.18}$$

By similarly reasons as in (3.8) and (3.9), from (3.16) and (3.17), we easily prove

$$\left\| g(.) - L_n(g,.) \right\|_{\upsilon,\langle\varphi\rangle,\frac{1}{n}} \le c\left(\frac{1}{n} \right)^r \left\| g^{(r)} \right\|_{\upsilon,\langle\varphi\rangle,\frac{1}{n}}, \quad g \in L_{\varphi,\pi}^{**,r+\upsilon} \tag{3.19}$$

$$\left\| L_n \right\|_{\upsilon,\langle\varphi\rangle} \le c, \quad n = 1,2,.... \tag{3.20}$$

Hence, for any $f \in L_{\varphi,\pi}^{**,r+\upsilon}$, by the corollary of Theorem 2.3, it follows that

$$\left\| f(.) - L_n(f,.) \right\|_{\upsilon,\langle\varphi\rangle,\frac{1}{n}} \le c\left(\frac{1}{n} \right)^{\upsilon} \omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)}, \frac{1}{n} \right).$$

Therefore, we can obtain

$$\left\| f^{(\upsilon)}(.) - L_n^{(\upsilon)}(f,.) \right\|_{\langle\varphi\rangle[-\pi,\pi]} \le c\omega_{r-\upsilon,\langle\varphi\rangle}\left(f^{(\upsilon)}, \frac{1}{n} \right).$$

4. Conclusion

In conclusion, in this paper we gave definition of another class of functions (Chen, 1964) that is wider than the classical Orlicz spaces L_{φ}^{*}. The wider class denoted by L_{φ}^{**} and the generating function φ of L_{φ}^{**} is not necessary to be convex (Chen, 1964). Then we proved some theorems on simultaneous approximation by trigonometric or algebraic polynomials in Orlicz spaces constructed by Young functions belonging to a reasonably wide class.

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تعريف آني بواسطة حدوديات على فضاءات أورلتز موكدة بواسطة دوال يونغ شبة المحدبة

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خلاصة

نقوم في هذا البحث بإثبات بعض النظريات على تقريب آنى بواسطة حدوديات جبرية أو مثلثية في فضاءات أورلتز و منشأة بواسطة دوال يونغ المنتمية إلى صنف كبير بشكل معقول.