Simultaneous approximation by polynomials in Orlicz spaces generated by quasiconvex Young functions

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Abstract

In this paper we prove some theorems on simultaneous approximation by trigonometric or algebraic polynomials in Orlicz spaces constructed by Young functions belonging to a reasonably wide class.

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1. Introduction

The problems of approximation by trigonometric/algebraic polynomials in classical Orlicz spaces were investigated by several mathematicians. Tsyganok (1966) obtained the Jackson type inequality of trigonometric approximation. Kokilashvili (1965) obtained inverse theorems of trigonometric approximation. Ponomarenko (1966) proved some direct theorem of trigonometric approximation by summation means of Fourier series. Cohen (1968) proved some direct theorem of trigonometric approximation by its partial sum of Fourier series. On the other hand simultaneous approximation of functions by trigonometric/algebraic polynomials in the classical Orlicz spaces were proved by Ramazanov (1984) and Garidi (1991). In these results the generating Young function of Orlicz spaces is convex. When the generating Young function satisfying quasiconvexity condition, similar problems were investigated in Akgün (2012), Akgün (2011), Akgün (2016), Akgün & Israfilov (2011) and Israfilov & Akgün (2010).

Present work deals with central problems of approximation by trigonometric/algebraic polynomials in Orlicz spaces having non convex generating Young functions. First of all we give basic definitions and notations.

Let \( \varphi : [0, \infty) \to [0, \infty) \) be a right continuous, monotone increasing function with \( \varphi(0) = 0; \lim_{t \to \infty} \varphi(t) = \infty \) and \( \varphi(t) > 0 \) whenever \( t > 0 \); then the function defined by

\[
\Phi(x) = \int_0^x f(t)dt
\]

is called \( N \)-function (Krasnosel’skii & Rutickii, 1961). The class of strictly increasing functions will be denoted by \( \Phi \). When \( \varphi \) is an \( N \)-function (Krasnosel’skii & Rutickii, 1961) we always denote by \( \psi(u) \) the mutually complementary \( N \)-function of \( \varphi(u) \). Let \( \varphi(u) \) be an \( N \)-function. We shall denote by \( L_\varphi \) the class of real-valued functions, defined on \( I := [a, b] \subset R \) such that

\[
\rho(u; \varphi) := \int_a^b \varphi\left[\frac{u(x)}{x}\right]dx < \infty.
\]

The classes \( L_\varphi \) are called Orlicz classes. The class of measurable functions \( f \) defined on \( I \) such that the product \( f(x)g(x) \) is integrable over \( (a, b) \) for every measurable function \( g \in L_\varphi \), will be denoted by \( L^*_\varphi \) which is called (classical) Orlicz space. We put

\[
\|f\| = \|f\|_\varphi := \sup_g \left| \int_a^b f(x)g(x)dx \right|
\]

where the supremum being taken with respect to all \( g \) with

\[
\rho_g := \rho(g; \varphi) = \int_a^b \psi\left[\frac{g(x)}{x}\right]dx \leq 1.
\]

Now we will give definition of another class of functions (Chen, 1964) that is wider than the classical Orlicz spaces \( L^*_\varphi \). The wider class will be denoted by \( L^*\varphi \) and the generating function \( \varphi \) of \( L^*\varphi \) is not necessary to be convex (Chen, 1964). We set \(-\infty < p \leq q < \infty \) and we...
denote by $Y[p,q]$ the class of even functions $\varphi \in \Phi$ defined on $(-\infty, \infty)$ satisfying the following two conditions

1. $\varphi(u)/u^p$ is non-decreasing when $|u|$ increases;
2. $\varphi(u)/u^q$ is non-increasing when $|u|$ increases.

When $p < q$ we will denote by $Y(p,q)$ the class of functions $\varphi$ satisfying $\varphi \in Y[p+\varepsilon, q-\delta]$ for some small numbers $\varepsilon, \delta > 0$.

The notation $\varphi(x) \sim [p_1, p_2]$, $0 \leq p_1 \leq p_2 \leq \infty$ (or similarly $-\infty \leq p_1 \leq p_2 \leq 0$) will indicate that for non-negative even function $\varphi(x)$, the function $\varphi(x) \cdot x^{-p}$ is non-decreasing and the function $\varphi(x) \cdot x^{-p}$ is non-increasing when $x$ is increasing in $(0, \infty)$.

Let $\varphi(x) \sim [p_1, p_2]$, $0 \leq p_1 \leq p_2 \leq \infty$ and $\varphi(t) := \varphi(t)/t$. We suppose that $\varphi_1(t) \to \infty$ as $t \to \infty$, and $\psi_1(t)$ be the inverse function of the positive non-decreasing continuous function $\varphi_1$. Defining as

$$\Phi_1(x) := \int_0^x \varphi_1(t)dt$$
$$\Psi_1(x) := \int_0^x \psi_1(t)dt$$

we get that $\Phi_1$ is a convex function and the functions $\Phi_1$, $\Psi_1$ are complementary functions. By $L_\varphi^*$ we will define the set of functions $f(x), a \leq x \leq b$, such that the product $f(x)g(x)$ is integrable over $(a, b)$ for any $g \in L_\varphi$. In $L_\varphi^*$ one can define a norm as

$$\|f\|_{(\varphi)} := \|f\|_{(\varphi)(t)} := \sup_g \int f(x)g(x)dx,$$

where the supremum being taken for all $g$ satisfying

$$\rho(g, \Psi_1) = \int_a^b |g(x)|dx \leq 1.$$  

Namely

$$L_\varphi^* := \left\{ f : [a,b] \to \mathbb{R} \mid \int_a^b f(x)g(x)dx < \infty \text{ for all } g \in L_\varphi \right\}.$$

For $f \in L_\varphi^*, g \in L_\varphi$, the generalized Hölder inequality (Chen, 1964)

$$\int_a^b f(x)g(x)dx \leq \|f\|_{(\varphi)} \|g\|_{(\psi_1)},$$

holds, where

$$\|g\|_{(\psi_1)} := \inf \left\{ k > 0 : \rho \left( g(x)/k, \Psi_1 \right) \leq 1 \right\}.$$

There are several important classes of $N-$functions. Among other things, these conditions relate to the growth of $N-$functions. Let $\varphi$ be an $N-$function. Then $\varphi$ is said to satisfy the $\Delta_2$ doubling condition (in notation: $\varphi \in \Delta_2$) (Chen, 1964); namely, there is a constant $C > 0$ and $x_0 > 0$ such that $\varphi(2x) \leq C\varphi(x)$ for all $x \geq x_0$. An Orlicz class is linear if and only if it satisfies the $\Delta_2$ doubling condition (Krasnosel’skii & Rutickii, 1961). Then the above defined function classes $L_{\varphi_1}, L_{\varphi_2}$ and $L_{\varphi_3}$ are identical (Chen, 1964). In general the class $L_{\varphi_1}^*$ is wider than $L_{\varphi_2}$ and $L_{\varphi_3}$ preserves the same properties of $L_{\varphi_1}^*$. In the class $L_{\varphi_1}^*$ the function $\varphi$ is not necessary to satisfy the convexity condition.

Ramazanov (1984) has obtained Jackson type theorem for the functions in Orlicz spaces $L_\varphi^*$. For further results see e.g. Akgün (2012), Akgün (2011), Akgün & Koç (2012) and Akgün & Koç (2016). Later Garidi (1991) extended the results of Ramazanov and proved Jackson type theorem for derivatives in the space

$$L_{\varphi_1}^*: \left\{ f \in L_{\varphi_1}^* \mid f^{(r)} \in L_{\varphi_1}^* \right\}.$$

But there are functions in $L_{\varphi_1}^*$ or $L_{\varphi_2}^*$ that does not belong to the class $L_{\varphi_1}^*$. For example taking as $\varphi \sim (2,3)$, $\varphi(x) = x^{5/2}$ for $0 \leq x \leq 1$, $\varphi(x) = x^{3/4}$ for $x > 1$ we have that (Chen, 1964) $\varphi$ is not a convex function. There exists (Chen, 1964) a method to find such functions. The main aim of this paper is to consider the simultaneous approximation by algebraic/trigonometric polynomials for functions in the Sobolev type space.

For $a = -\pi$ and $b = \pi$ then we will use the notations $L_{\varphi_1}^*$ and $L_{\varphi_2}^*$. For the simplicity every where in this work, the constant $c$ will denote different positive real number in different places.

$$L_{\varphi_1}^* := \left\{ f : [a,b] \to \mathbb{R} \mid \int_a^b f(x)g(x)dx < \infty \text{ for all } g \in L_{\varphi_1} \right\}.$$

For any $f \in L_{\varphi_1}^*$ there exists an algebraic polynomial $P$ of degree $n$ such that

$$f(x) = P(x) + \int P(t)dt,$$

holds for any integer $n \geq 1$ where $c$ is some constant depending only on $r$ and $\varphi$.

Theorem 1.1. Let $1 < p < q < \infty, \varphi \in Y(p,q), r = 1, 2, 3, \ldots$ and $\nu = 0, 1, 2, \ldots, r$. For any $f \in L_{\varphi_1}^*$ there exists an algebraic polynomial $P$ of degree $n$ such that

$$\|f^{(\nu)} - P^{(\nu)}\|_{(\varphi)} \leq c\omega_{r}\nu, \varphi, (\nu) \left( f^{(\nu)}, 1/n \right)$$

holds for any integer $n \geq 1$ where $c$ is some constant depending only on $r$ and $\varphi$.

Theorem 1.2. Let $1 < p < q < \infty, \varphi \in Y(p,q), r = 1, 2, 3, \ldots$ and $\nu = 0, 1, 2, \ldots, r$. For any $f \in L_{\varphi_1}^*$ there exists a trigonometric polynomial $T$ of degree $n$ such that
holds for any integer \( n \geq 1 \) where \( c \) is some constant depending only on \( r \) and \( \varphi \).

In these theorems \( \omega_{r,(\varphi)} \) denotes \( r \)-th modulus of smoothness which given in (2.3).

2. \( K \)-Functional and modulus of smoothness in \( L^*_q \)

Suppose that \( r = 1, 2, 3, \ldots, \nu = 0, 1, 2, \ldots, r, t > 0 \) and \( f \in L^*_q \). We define

\[
\| f \|_{r,(\varphi),t} = \| f \|_{r,(\varphi)} := \sum_{i=0}^{\nu} t^i \| f^{(i)} \|_{(\varphi)}
\]  

(2.1)

and \( K \)-functional by

\[
K^v_{r,(\varphi)} (f,t) := \inf \{ \| f - g \|_{r,(\varphi)} + t^r \| g^{(r)} \|_{r,(\varphi)} : g \in L^*_q \}. 
\]  

(2.2)

For \( h \geq 0 \) and

\[
I_h := \begin{cases} \{ [a,b-h] \}, & 0 \leq h < b-a \\ \emptyset, & h > b-a \end{cases}
\]

the expression

\[
\Delta'_r (f,x) := \sum_{i=0}^{r} (-1)^i \binom{r}{i} f \left( x + it \right)
\]

is called \( r \)-th difference of the function \( f \).

For the empty set \( \emptyset \), we define \( \| f \|_{(\varphi)(\emptyset)} = 0 \). For \( f \in L^*_q \), we define its \( r \)-th modulus of smoothness as

\[
\omega_{r,(\varphi)} (f,t) = \sup_{0 \leq s < t} \| \Delta'_r (f,s) \|_{(\varphi)(I_r)}. 
\]  

(2.3)

Remark 2.1. The modulus \( \omega_{r,(\varphi)} (f,t) \) possesses the following usual properties:

1. \( \omega_{r,(\varphi)} (f,t) \) is a monotone increasing function of \( t \) and \( \omega_{r,(\varphi)} (f,0) = 0 \).
2. For any \( f \in L^*_q \), \( \omega_{r,(\varphi)} (f,t) \to 0 \) as \( t \to 0 \) iff \( \varphi \) satisfies the \( \Delta_2 \)-condition.
3. If \( f \in L^*_q \), then \( \omega_{r,n,(\varphi)} (f,t) \leq t^n \omega_{r,(\varphi)} (f(t),t) \).
4. \( \omega_{r,(\varphi)} (f,nt) \leq n^n \omega_{r,(\varphi)} (f,t) \) for any non-negative integer \( n \).

Theorem 2.2. The function \( \varphi \) satisfies the \( \Delta_2 \)-condition if and only if “for given any positive integer \( r \), \( \omega_{r,(\varphi)} (f,t) \to 0 \) holds true as \( t \to 0 \) for every \( f \in L^*_q \).”

Proof of Theorem 2.2. Suppose that \( \varphi \) satisfies the \( \Delta_2 \)-condition and that \( P_n \) \((n = 1, 2, \ldots)\) is a sequence of polynomials which convergence to \( f \in L^*_q \). As in Garidi (1991) for \( 0 < h < \frac{b-a}{r} \) we have

\[
\| \Delta^h (f-P_n,x) \|_{[a,b-h]} = \sup_{g \in L_{q_1}} \left\{ \int_a^{b-h} \left| \Delta^h (f-P_n,x) g(x) \right| dx : \rho(g,\Psi_1) \leq 1 \right\}
\]

\[
= \sup \left\{ \int_a^{b-h} \sum_{i=0}^{r} (-1)^i \binom{r}{i} (f-P_n) \left( x + ih \right) g(x) dx : g \in L_{q_1}, \rho(g,\Psi_1) \leq 1 \right\}
\]

\[
\leq \sup \left\{ \sum_{i=0}^{r} \int_a^{b-h} \left| (f-P_n) \left( x + ih \right) g(x) \right| dx : g \in L_{q_1}, \rho(g,\Psi_1) \leq 1 \right\}
\]

\[
= \sup \left\{ \sum_{i=0}^{r} \int_a^{b-h} \left| (f-P_n) \left( x + ih \right) \right| dx : g \in L_{q_1}, \rho(g,\Psi_1) \leq 1 \right\}
\]

\[
\leq \sum_{i=0}^{r} \left\{ \sup_{g \in L_{q_1}} \int_a^{b-h} \left| (f-P_n) \left( x + ih \right) g(x) \right| dx : g \in L_{q_1}, \rho(g,\Psi_1) \leq 1 \right\}
\]

\[
= \sum_{i=0}^{r} \left\{ \int_a^{b-h} \left| (f-P_n) \left( x + ih \right) \right| dx : g \in L_{q_1}, \rho(g,\Psi_1) \leq 1 \right\}.
\]
Since the space is invariant under translation, writing \( x + ih = u \) and \( 0 \leq i \leq r, a \leq x \leq br - h, a \leq x + rh \leq b \) we obtain

\[
\left\| \Delta^h_r (f, \cdot) \right\|_{\mathcal{L}^\phi[a, b - rh]} = \left\| \Delta^h_r (f - P_n, \cdot) + \Delta^h_r (P_n, \cdot) \right\|_{\mathcal{L}^\phi[a, b - rh]} \\
\leq \left\| \Delta^h_r (f - P_n, \cdot) \right\|_{\mathcal{L}^\phi[a, b - rh]} + \left\| \Delta^h_r (P_n, \cdot) \right\|_{\mathcal{L}^\phi[a, b - rh]} \\
\leq 2^r \left\| f - P_n \right\|_{\mathcal{L}^\phi[a, b - rh]} + \left\| \Delta^h_r (P_n, \cdot) \right\|_{\mathcal{L}^\phi[a, b - rh]}.
\]

Therefore we have

\[
\left\| \Delta^h_r (f, \cdot) \right\|_{\mathcal{L}^\phi[a, b - rh]} \leq 2^r \left\| f - P_n \right\|_{\mathcal{L}^\phi[a, b - rh]} + \left\| \Delta^h_r (P_n, \cdot) \right\|_{\mathcal{L}^\phi[a, b - rh]} < \varepsilon.
\]

Thus, \( \omega_r (\phi) \rightarrow 0 \), as \( n \rightarrow \infty \) since

\[ c \omega_r (f) < \varepsilon. \]

The opposite way of Theorem 2.2 is easy.

Suppose \( E \subseteq L^{r+\nu} \), for suitable \( r > 0 \), we introduce the following best degree of approximation

\[
\rho_{r, \phi} (f, E) = \inf_{g \in E} \left\| f - g \right\|_{r, \phi}. \tag{2.4}
\]

Theorem 2.3. Suppose that \( r = 0, 1, \ldots, \nu = 0, 1, \ldots, r, t > 0 \) and \( f \in L^{r+\nu} \). Then

\[
c_t^r \omega_{r, \nu, \phi} (f^{(i)}, t) \leq K_{r, \phi} (f, t) \leq c_t^r \omega_{r, t, \phi} (f^{(i)}, t). \tag{2.5}
\]

where the constants \( c_1 \) and \( c_2 \) are depending only on \( r \) and \( \phi \).

Proof of Theorem 2.3. First, let us prove the lower estimate. We will follow the method given in (Garidi, 1991). For \( 0 \leq \nu \leq r, f \in L^{r+\nu} \) and any \( g \in L^{r+\nu} \), from the property (3) of \( \omega_{r, \nu, \phi} (f, t) \), we obtain

\[
\omega_{r, \nu, \phi} (f^{(i)}, t) \leq \omega_{r, \nu, \phi} (f^{(i)} - g, t) + \omega_{r, \nu, \phi} (g, t) \\
\leq 2^{-\nu} \omega_{r, \nu, \phi} (f^{(i)} - g, t) + 2^{-\nu} \omega_{r, \nu, \phi} (g, t) \\
\leq c \left( \left\| f^{(i)} - g \right\|_{\phi} + t^{r-\nu} \left\| g^{(r-\nu)} \right\|_{\phi} \right).
\]

Since \( g \) is arbitrary we have

\[
c_t^r \omega_{r, \nu, \phi} (f^{(i)}, t) \leq t^r K_{r, \nu, \phi}^{0} (f^{(i)}, t). \tag{2.6}
\]

and for \( f \in L^{r+\nu} \), \( i = 0, 1, \ldots, \nu - 1 \) there holds

\[
K_{r, \nu, \phi}^{i} (f^{(i)}, t) = \inf_{h \in L^{r+\nu}} \left\{ \left\| f^{(i)} - g \right\|_{r, \phi} + t^{i+\nu} \left\| g^{(r-\nu)} \right\|_{r, \phi} : g \right\} \\
\geq \inf \left\{ t \left\| f^{(i)} - g \right\|_{r, \phi} + t^{i+\nu} \left\| h^{(r-\nu)} \right\|_{r, \phi} : g \right\} \\
\geq t. \inf \left\{ t \left\| f^{(i)} - h \right\|_{r, \phi} + t^{i+\nu} \left\| h^{(r-\nu)} \right\|_{r, \phi} : h \right\}
\]

From this recurrence formula, we obtain

\[
K_{r, \nu, \phi}^{i} (f, t) \geq t^r K_{r, \nu, \phi}^{0} (f^{(i)}, t). \tag{2.8}
\]

Thus, from (2.6) and (2.8), it follows the lower estimate.

\[
c_t^r \omega_{r, \nu, \phi} (f^{(i)}, t) \leq K_{r, \nu, \phi}^{i} (f, t).
\]
Suppose $\int_{I_0} \Psi_i(v(x)) dx \leq 1$. Then for $i = 0, 1, \ldots, r$, we have

$$\left| \int_{I_0} (f^{(i)}(x) - g^{(i)}_0(x)) v(x) dx \right| = \left| \int_{I_0} \left( t' \int_{I_0} \Delta^{i}_{\eta_{1}+\ldots+\eta_{r}}(f^{(i)},x) du_{1} \ldots du_{r} \right) v(x) dx \right|$$

$$= \left| t' \int_{I_0} \left( \int_{I_0} \Delta^{i}_{\eta_{1}+\ldots+\eta_{r}}(f^{(i)},x) v(x) dx \right) du_{1} \ldots du_{r} \right|$$

$$\leq t' \int_{I_0} \left( \sum_{j=1}^{r} (-1)^{r-j} \left( r \right)^{j} \Delta^{i}_{\eta_{1}+\ldots+\eta_{r}}(f^{(i)},x) \right) v(x) dx$$

$$= \omega_{r,\eta}(f^{(i)},r) \leq \omega_{r,\eta}(f^{(i)},t).$$

Hence

$$\left\| f^{(i)} - g^{(i)}_0 \right\|_{(\omega_{r,\eta}(t_0))} \leq c t'^{\nu-\nu} \omega_{r-\nu,\eta}(f^{(\nu)},t). \quad (2.9)$$

By similar arguments, we get

$$t' \left\| g^{(r+i)}_0 \right\|_{(\omega_{r,\eta}(t_0))} \leq \sum_{j=1}^{r}(-1)^{r-j} \left( r \right)^{j} \Delta^{i}_{\eta_{1}+\ldots+\eta_{r}}(f^{(i)},x) \leq c t'^{\nu-\nu} \omega_{r-\nu,\eta}(f^{(\nu)},t). \quad (2.10)$$

Let us take a function $\varphi(x)$ with $\varphi(x) = 0$ for $x \in [a, a + \frac{b-a}{2}]$, $\varphi(x) = 1$ for $x \in [a + \frac{b-a}{2}, b]$ and $\left| \varphi^{(j)}(x) \right| \leq c$ for $i = 0, 1, \ldots, 2r$, $x \in [a, b]$. We extend $g_0 \in L^{*}_{\varphi}(I_0)$ and $g_i \in L^{*}_{\varphi}(I_i)$ according to $g_0, g_i \in L^{*}_{\varphi}$ (This is possible by a method of Ramazanov (1984)). Let

$$g(x) := (1 - \varphi(x)) g_0(x) + \varphi(x) g_1(x). \quad (2.13)$$

Then, for $i = 0, 1, \ldots, r + \nu$, we have

$$g^{(i)}(x) = \left( 1 - \varphi(x) \right) g^{(i)}_0(x) + \varphi(x) g^{(i)}_1(x) + \sum_{j=0}^{i-1} \varphi^{(i-j)}(x) \left( g^{(j)}_1(x) - g^{(j)}_0(x) \right) \quad (2.14)$$

hence, for $i = 0, 1, \ldots, r + \nu$, from (2.9), (2.10) and (2.14), we have

$$\left\| f^{(i)} - g^{(i)}_0 \right\|_{(\omega_{r,\eta}(t_0))} \leq \left\| f^{(i)} - g^{(i)}_1 \right\|_{(\omega_{r,\eta}(t_0))} \leq c t'^{\nu-\nu} \omega_{r-\nu,\eta}(f^{(\nu)},t), \quad (2.15)$$

$$\left\| f^{(i)} - g^{(i)}_0 \right\|_{(\omega_{r,\eta}(t_0))} \leq \left\| f^{(i)} - g^{(i)}_0 \right\|_{(\omega_{r,\eta}(t_0))} \leq c t'^{\nu-\nu} \omega_{r-\nu,\eta}(f^{(\nu)},t), \quad (2.16)$$
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For \( i = 0, 1, \ldots, r \) from (2.9) to (2.14) and Lemma 1a in Ramazanov (1984), we deduce

\[
\|f^{(i)} - g^{(i)}\|_{\omega_{r(u,\phi)}} \leq \|f^{(i)} - g_0^{(i)}\|_{\omega_{r(u,\phi)}} + \|f^{(i)} - g_1^{(i)}\|_{\omega_{r(u,\phi)}} + c \sum_{j=0}^{i} \|f^{(j)} - g_0^{(j)}\|_{\omega_{r(u,\phi)}} + \|f^{(j)} - g_1^{(j)}\|_{\omega_{r(u,\phi)}}.
\]

Hence using \((2.15)\) to \((2.18)\), for \( f^{(i)} \) we have

\[
\|f^{(i)} - g_0^{(i)}\|_{\omega_{r(u,\phi)}} + \|f^{(i)} - g_1^{(i)}\|_{\omega_{r(u,\phi)}} + \|f - g_0\|_{\omega_{r(u,\phi)}} + \|f - g_1\|_{\omega_{r(u,\phi)}} 
\leq \|f - g_0\|_{\omega_{r(u,\phi)}} + \|f - g_1\|_{\omega_{r(u,\phi)}} + \|f^{(i)} - g_1^{(i)}\|_{\omega_{r(u,\phi)}} + \|f^{(i)} - g_0^{(i)}\|_{\omega_{r(u,\phi)}} + \|f^{(i)} - g\|_{\omega_{r(u,\phi)}} + \|f^{(i)} - g\|_{\omega_{r(u,\phi)}}
\]

Thus, for \( 0 < t \leq b - a \), we have

\[
K_{r,\phi}^u(f, st) \leq \|f - g\|_{\omega_{r(u,\phi)}} + t \|g^{(i)}\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} + t \|g^{(i)}\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} + t \|g^{(i)}\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} + t \|g^{(i)}\|_{\omega_{r(u,\phi)}}
\]

\[
K_{r,\phi}^u(f, st) \leq s^{r+u} c t \|f - g\|_{\omega_{r(u,\phi)}} + t \|g^{(i)}\|_{\omega_{r(u,\phi)}} \leq s^{r+u} c t \|f - g\|_{\omega_{r(u,\phi)}} + t \|g^{(i)}\|_{\omega_{r(u,\phi)}} \leq s^{r+u} c t \|f - g\|_{\omega_{r(u,\phi)}} + t \|g^{(i)}\|_{\omega_{r(u,\phi)}}
\]

Thus, for \( 0 < t \leq b - a \), we have

\[
K_{r,\phi}^u(f, st) \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}}
\]

Lemma 2.4. Suppose that \( r = 1, 2, \ldots, u \) and \( v = 0, 1, \ldots, r \), then for any \( f \in L^r_{\phi} \) and \( t \geq b - a \), we have

\[
\rho_{r,\phi}(f, P_{r-1}) \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}} \leq c t \|f - g\|_{\omega_{r(u,\phi)}}
\]

where \( c \) is a constant depending on \( r \) only. Obviously, \( P_{r-1} \subset L^r_{\phi} \) and for any \( p \), \( i = 0, 1, \ldots, u \), \( p^{(r)}(x) = 0 \). Hence, for \( r \geq 1 \), \( K_{r,\phi}^u(f, st) \leq \rho_{r,\phi}(f, P_{r-1}) \).

Thus, by Lemma 2.4 given in below, we can easily prove that Theorem 2.3 is also true for \( t \geq b - a \). Now the proof of Theorem 2.3 is completed.
Like in Garidi (1991) we set \( p(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}}{i!}(x-a)\) \( i \in P_{r+1} \). Then for \( i = 1, \ldots, r \) and \( x \in [a, b] \), we have

\[
\left| g^{(i-1)}(x) - p^{(i-1)}(x) \right| = \left| \int_{a}^{x} \left( g^{(i)}(t) - p^{(i)}(t) \right) dt \right| \\
\leq \int_{a}^{x} \left| g^{(i)}(t) - p^{(i)}(t) \right| dt \leq \left\| g^{(i)} - p^{(i)} \right\|_{\varphi} \left\| \varphi \right\|_{\varphi}.
\]

Thus, for \( i = 1, \ldots, r \), it is easy to see that

\[
\left\| g^{(i-1)} - p^{(i-1)} \right\|_{\varphi} = \sup_{\varphi(\psi,\bar{\psi}) \in [a, b]} \left| \int_{a}^{b} \left( g^{(i-1)}(x) - p^{(i-1)}(x) \right) \psi(x) dx \right| \\
\leq \left\| g^{(i)} - p^{(i)} \right\|_{\varphi} \left\| \psi \right\|_{\varphi} \sup_{\varphi(\psi,\bar{\psi}) \in [a, b]} \int_{a}^{b} \psi(x) dx \\
= \left\| g^{(i)} - p^{(i)} \right\|_{\varphi} \leq c(b-a) \left\| g^{(i)} - p^{(i)} \right\|_{\varphi}.
\]

Hence, for \( t \geq b-a \), we can obtain

\[
\left\| f - p \right\|_{\varphi, t} \leq \left\| f - g \right\|_{\varphi, t} + \left\| g - p \right\|_{\varphi, t} \\
\leq \left\| f - g \right\|_{\varphi, t} + \left\| (b-a)^{p} \sum_{i=0}^{\nu} \left( \frac{t}{c(b-a)} \right)^{i} \right\|_{\varphi} \\
\leq c \left\{ \left\| f - g \right\|_{\varphi, t} + \left\| (b-a)^{\nu} \right\|_{\varphi} \sum_{i=0}^{\nu} \left( \frac{t}{c(b-a)} \right)^{i} \right\} \\\n\leq c \left( \frac{t}{b-a} \right)^{\nu} \left\{ \left\| f - g \right\|_{\varphi, t} + \left\| (b-a)^{\nu} \right\|_{\varphi} \left( \nu + 1 \right) \right\} \\
\leq c \left( \frac{t}{b-a} \right)^{\nu} \left\{ \left\| f - g \right\|_{\varphi, t} + (b-a)^{\nu} \right\|_{\varphi} \left( \nu + 1 \right), \right\}
\]

In view of (2.21), (2.22) and the following inequality

\[
(b-a)^{\nu} \left\| \varphi \right\|_{\varphi} \left( f^{(\nu)}, b-a \right) \leq t^{\nu} \left\| \varphi \right\|_{\varphi} \left( f^{(\nu)}, t \right)
\]

for \( t \geq b-a \), we complete the proof of Lemma 2.4.

Corollary 2.5. Suppose that L is a bounded linear operator from \( L_{x}^{\nu} \) to \( L_{x}^{\nu} \) and \( \left\| L \right\|_{\varphi, \psi} \leq c_{\psi} \). If for any \( g \in L_{x}^{\nu} \) the inequality

\[
\left\| g - L(g) \right\|_{\varphi, \psi} \leq c_{\psi} \left\| \varphi \right\|_{\varphi} \left( f^{(\nu)}, t \right)
\]

holds, then for any \( f \in L_{x}^{\nu} \), we have

\[
\left\| f - L(f) \right\|_{\varphi, \psi} \leq c_{\psi} \left\| \varphi \right\|_{\varphi} \left( f^{(\nu)}, t \right),
\]

where \( L_{\varphi, \psi} = \sup_{L \in \mathcal{L}_{\varphi, \psi}} \left\| L(f) \right\|_{\varphi, \psi} \) and \( c \) depends on \( r \) and \( \varphi \) only.

3. Proof of the main results

For the proof of the main results we will need the following Lemmas.

Lemma 3.1. Suppose that \( I = [a, b] \) and \( J = [c, d] \) are closed intervals such that \( I \subset J \), \( r = 0, 1, \ldots \), and \( f(x) \) is a function defined on \( I \). We extend \( f(x) \) from \( I \) to \( J \) by the following formula

\[
(f(x), t) = \left\{ \begin{array}{ll}
\left( f(x), t \right) & \text{if } x \in [a, b], \\
\left( f((x-a)/r), (x-a)/r \right) & \text{if } x \in [b, c], \\
\left( f((x-b)/r), (x-b)/r \right) & \text{if } x \in [c, d].
\end{array} \right.
\]
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where \( \{ \alpha_i \} \) and \( \{ \beta_i \} \) are real numbers which satisfy the following conditions

\[
\sum_{i=0}^{2^r} \alpha_i (-2^{-i \frac{a}{2^r}} t)^j = 1, \quad \sum_{i=0}^{2^r} \beta_i (-2^{-i \frac{b}{2^r}} t)^j = 1, \quad j = 0, 1, \ldots, 2^r.
\]

Then for \( i, j = 0, 1, \ldots, r, t > 0 \) and any \( f \in L_{\psi}^{r, i} \), we have

\[
\omega_{j,r}(f) \left( t \right) \leq c \omega_{j,r}(f) \left( f^{(i)}(t) \right),
\]

where \( c \) depends on \( r, I \) and \( J \).

The proof of Lemma 3.1 can be done by using the method of that Lemma in Ramazanov (1984) and Theorem 2 (Ramazanov, 1984). Let \( P_n \) be the set of all algebraic polynomials of degree \( n \).

We take \( \lambda_n(t) \in P_n \) such that

\[
\frac{1}{4^r} \int_{-1}^{1} \lambda_n(t) dt = 1. \quad (3.1)
\]

Let

\[
K_n(t) := \lambda_n \left( \frac{t}{4} \right) / 4, \quad (3.2)
\]

\[
\|
\| f^{(i)}(\cdot) - \Phi_n^{(i)}(f, \cdot) \|_{L_{\psi}^{r, i}} \leq c \left\{ \omega_{r, \nu, \psi}(f^{(i)}) \left( f^{(i)} \right), n^{1-4r+\nu} \left\| f^{(i)} \right\|_{L_{\psi}^{r, i}} \}
\]

where \( c \) depends on \( r \) only.

Proof of Lemma 3.3. Following the arguments in Garidi (1991) let

\[
E_{t, x} = \left[ \frac{-2-x}{i}, \frac{2-x}{i} \right], \quad R_{t, x} = \int_{E_{t, x}} f_0(x + it) K_n(t) dt
\]

then, we have

\[
\Phi_n(f, x) = \int_0^1 \left\{ f_0(x) + (-1)^{r+1} \Delta_i(f_0, x) \right\} K_n(t) dt + \sum_{i=1}^r (-1)^{i+1} \left( \frac{r}{i} \right) R_i(x)
\]

= \Phi_n(f, x) + B_n(f, x).
Therefore
\[ \| f^{(i)}(\cdot) - \Phi_n^{(i)}(f, \cdot) \|_{(\varphi)} \leq \| f^{(i)}(\cdot) - A_n^{(i)}(f, \cdot) \|_{(\varphi)} + \| B_n^{(i)}(f, \cdot) \|_{(\varphi)}. \]

Assume that \( g \in L^r_{\varphi} \) and \( g_0 \) is the extended function of \( g \) from \([-1, 1]\) to \([-2, 2]\). Then, it follows that from (3.1),
\[ g^{(i)}(x) - A_n^{(i)}(g, x) = (-1)^i \frac{1}{r} \int_{-1}^{1} \Delta_t^{i} \left( g_0^{(i)}, x \right) K_n(t) dt, \quad i = 0, 1, \ldots, \nu. \]

For \( i = 0, 1, \ldots, \nu \), from Lemma 3.1 and (3.6), we have
\[
\| g^{(i)}(\cdot) - A_n^{(i)}(g, \cdot) \|_{(\varphi)} = \sup_{\rho(\nu, \varphi_r) \leq 1} \left| \int_{-1}^{1} \Delta_t^{i} \left( g_0^{(i)}, x \right) K_n(t) dt \right| \nu(x) dx
\]
\[
\leq \int_{-1}^{1} \frac{1}{r} \left( \sup_{\rho(\nu, \varphi_r) \leq 1} \left| \int_{-1}^{1} \Delta_t^{i} \left( g_0^{(i)}, x \right) \nu(x) dx \right| \right) K_n(t) dt
\]
\[
\leq \int_{-1}^{1} \frac{1}{r} \omega_r(\varphi_r) \left( g_0^{(i)}, t \right) K_n(t) dt \leq \frac{1}{r} \int_{-1}^{1} \left\| g_0^{(i+r)} \right\|_{[\varphi[-2,2]}} K_n(t) dt
\]
\[
\leq c \left\| g^{(i+r)} \right\|_{[\varphi[-1,1]}} \int_{-1}^{1} \left| t \right|^r K_n(t) dt \leq cn^{-r} \left\| g^{(i+r)} \right\|_{[\varphi[-1,1]}.
\]

Hence, for \( t = \frac{1}{n} \) and any \( g \in L^r_{\varphi} \), we have
\[
\left\| g(\cdot) - A_n(g, \cdot) \right\|_{\omega(\varphi, \frac{1}{n})} \leq c \left( \frac{1}{n} \right)^r \left\| g^{(r)} \right\|_{\omega(\varphi, \frac{1}{n})}.
\]

On the other hand, for \( f \in L^r_{\varphi} \) and \( i = 0, 1, \ldots, \nu \), we easily get
\[
\left\| A_n^{(i)}(f, \cdot) \right\|_{(\varphi)} = \left\| f^{(i)}(x) + \sum_{j=0}^{r} \frac{1}{r} \int_{-1}^{1} (-1)^{r-j} \left( \frac{r}{j} \right) f_0^{(i)}(x + jt) K_n(t) dt \right\|_{(\varphi)}
\]
\[
\leq \left\| f^{(i)}(x) \right\|_{(\varphi)} + \sup_{\rho(\nu, \varphi_r) \leq 1} \left\| \int_{-1}^{1} \sum_{j=0}^{r} (-1)^{r-j} \left( \frac{r}{j} \right) f_0^{(i)}(x + jt) K_n(t) dt \right\| \nu(x) dx
\]
\[
\leq \left\| f^{(i)}(x) \right\|_{(\varphi)} + \sum_{j=0}^{r} \left( \frac{r}{j} \right) \int_{-1}^{1} \left\| \sup_{\rho(\nu, \varphi_r) \leq 1} \int_{-1}^{1} f_0^{(i)}(y) \nu(y - j\eta) dy \right\| K_n(t) dt
\]
Thus, we obtain

\[
\|A_n(f, \cdot)\|_{\varphi, \langle \varphi \rangle} \leq c\|f\|_{\varphi, \langle \varphi \rangle}, \quad f \in L^{**}[-1,1]
\]

which implies

\[
\|A_n\|_{\varphi, \langle \varphi \rangle} \leq c, \quad (n = 1, 2, \ldots).
\]

(3.8), (3.9) and from the corollary of Theorem 2.3, for \( f \in L^{**}[-1,1] \)

\[
\|f(\cdot) - A_n(f, \cdot)\|_{\varphi, \langle \varphi \rangle} \leq c\left(\frac{1}{n}\right)^\varphi \omega_{r, \varphi, \langle \varphi \rangle}\left(f^{(u)}(\cdot), \frac{1}{n}\right).
\]

Therefore, it follows that

\[
\left\|f^{(u)}(\cdot) - A_n^{(u)}(f, \cdot)\right\|_{\langle \varphi \rangle} \leq c\omega_{r, \varphi, \langle \varphi \rangle}\left(f^{(u)}(\cdot), \frac{1}{n}\right).
\]

(3.10)

In addition, in view of

\[
B_n^{(v)}(f, x) = \sum_{i=0}^{r}(-1)^{i+1}\binom{r}{i}R_i^{(v)}(x)
\]

and

\[
R_i^{(v)}(x) = \int_{E_{i,x}} f_0^{(v)}(x + it)K_n(t)dt + \sum_{j=0}^{i-1}\left(1 - \frac{1}{i}\right)^{j+1}\left[f_0^{(v-1-j)}(2)K_n^{(j)}\left(\frac{2-x}{i}\right) - f_0^{(v-1-j)}(-2)K_n^{(j)}\left(-\frac{2-x}{i}\right)\right]
\]

from (3.7), we can obtain

\[
\left\|\int_{E_{i,x}} f_0^{(v)}(x + it)K_n(t)dt\right\|_{\varphi, \langle \varphi \rangle} = \sup_{\rho(\varphi, \langle \varphi \rangle) = 1} \int_{E_{i,x}} \left\|\int_{E_{i,x}} f_0^{(v)}(x + it)K_n(t)dt\right\|_{\varphi}v(x)dx
\]

\[
\leq \int_{E_{i,x}} \left\|\sup_{\rho(\varphi, \langle \varphi \rangle) = 1} \left\|\int_{E_{i,x}} f_0^{(v)}(y)\psi(y - it)dy\right\|_{\varphi}K_n(t)dt
\]

\[
\leq c\left\|f_0^{(v)}\right\|_{\langle \varphi \rangle} K_n(t)dt \leq cn^{-4r}\left\|f^{(v)}\right\|_{\langle \varphi \rangle}^{-1,1,2} \leq cn^{-4r}\left\|f^{(v)}\right\|_{\langle \varphi \rangle}.
\]

(3.11)
Thus, by the assumptions, we have that $f^{(i)}$ have zeros on $[-1,1]$ for $i = 1,2,...,\nu - 1$.

Suppose that the zeros of $f^{(i)}$ are $c_i$ for $i = 1,2,...,\nu - 1$. Then, for $x \in [-2,2]$, we have

$$|f^{(i)}_0(x)| = \left| \int_{-2}^{2} f^{(i+1)}(t) \, dt \right| \leq \int_{-2}^{2} |f^{(i+1)}_0(x)| \, dx$$

$$\leq \left\| f^{(i+1)}_0 \right\|_{[\phi][-2,2]} \cdot \left\| \phi \right\|_{L^1} \leq c \left\| f^{(i+1)}_0 \right\|_{[\phi][-1,1]}.$$

Hence, it follows that for $i = 1,2,...,\nu - 1$

By using (3.7) again, we get

$$\sum_{j=0}^{\nu - 1} \left( \frac{-1}{i} \right)^{j+1} \left[ f^{(i-j)}_0 (2) K^{(j)}_n \left( \frac{2-x}{i} \right) - f^{(i-j)}_0 (-2) K^{(j)}_n \left( \frac{-2-x}{i} \right) \right] \leq c n^{1-4r+\nu} \left\| f^{(i)}_0 \right\|_{[\phi]}.$$

Hence, we have

$$\left\| \sum_{j=0}^{\nu - 1} \left( \frac{-1}{i} \right)^{j+1} \left[ f^{(i-j)}_0 (2) K^{(j)}_n \left( \frac{2-x}{i} \right) - f^{(i-j)}_0 (-2) K^{(j)}_n \left( \frac{-2-x}{i} \right) \right] \right\|_{[\phi]} \leq c n^{1-4r+\nu} \left\| f^{(i)}_0 \right\|_{[\phi]}.$$

(3.12)

From (3.11) and (3.12), we obtain

$$\left\| K^{(i)}_n \right\|_{[\phi]} \leq c n^{1-4r+\nu} \left\| f^{(i)}_0 \right\|_{[\phi]}, \quad i = 1,\ldots,r.$$

Therefore,

$$\left\| B^{(i)}_n \right\|_{[\phi]} \leq c n^{1-4r+\nu} \left\| f^{(i)}_0 \right\|_{[\phi]}.$$

(3.13)

From (3.10) and (3.13), we can complete the proof of Lemma 3.3.

Lemma 3.4. Suppose that $r = 1,2,...,\nu = 0,1,...,\nu$ and $q(f,\cdot)$ is the same as above.

Let $\left\| q^{(i)} \right\| = \sup_{f \in L^\nu_{\phi}} \left\| q^{(i)}(f,\cdot) \right\|_{[\phi]}$. Then, for any $f \in L^\nu_{\phi}$, we have

$$\left\| f^{(i)}(\cdot) - q^{(i)}(f,\cdot) \right\|_{[\phi]} \leq c \left( 1 + \left\| q^{(i)} \right\|_{[\phi]} \right) (b-a)^{\nu} \omega_{r,v,\phi}(f^{(i)},b-a)$$

where $c$ depends on $r$ only.

Proof of Lemma 3.4. Assume that $Q(x) \in P_{r+1}$. Then, by using Lemma 2.4, we obtain (Garidi, 1991)

$$\left\| f^{(i)}(\cdot) - q^{(i)}(f,\cdot) \right\|_{[\phi]} \leq \left\| f^{(i)} - Q^{(i)} \right\|_{[\phi]} + \left\| q^{(i)}(f-Q,\cdot) \right\|_{[\phi]}$$
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Proof of Theorem 1.2. From Chapter 4 in Lorentz (1966), it follows that for \( x \in [-1,1] \), there exists \( T_n \) such that \( T_n \) is a trigonometric polynomial of degree \( n \), we have

\[
q^{(i)}(f,x) \leq c \max_{x \in [-1,1]} q(f,x) = c \max_{x \in [-1,1]} Q_i(x) \leq c \max_{x \in [-1,1]} Q_0(x). \tag{3.16}
\]

By using Lemma 4 in Yiqun (1984), we obtain

\[
c \max_{x \in [-1,1]} |Q_0(x)| \leq c \max_{x \in [-1,1]} F(x) = c \max_{x \in [-1,1]} \int_0^x f(t) dt \]

\[
\leq c \int_{-1}^{1} |f(x)| dx \leq c \|f\|_{(\varphi)} \|f\|_{(\varphi)} \leq c \|f\|_{(\varphi)}. \tag{3.17}
\]

From (3.14) and (3.15), it follows that

\[
\|q^{(i)}(f)\|_{(\varphi)} \leq c \|f\|_{(\varphi)}, \quad \|q^{(i)}\|_{(\varphi)} \leq c.
\]

By Lemma 3.3, Lemma 3.4 and notice that \( q(f,x) \) is a polynomial of degree \( r-1 \), (see for example Garidi, 1991) we have

\[
\|f^{(i)}(\cdot) - L_n^{(i)}(\cdot)\|_{(\varphi)} = \|f^{(i)}(\cdot) - q^{(i)}(\cdot) - \Phi_n^{(i)} f - q(f)\|_{(\varphi)}
\]

\[
\leq c \left\{ \omega_{r-\nu}(\varphi) \left( f^{(i)}(\cdot) \frac{1}{n} + n^{1-4r+\nu} \|f^{(i)}(\cdot) - q^{(i)}(\cdot)\|_{(\varphi)} \right) \right\}
\]

\[
\leq c \left\{ \omega_{r-\nu}(\varphi) \left( f^{(i)}(\cdot) \frac{1}{n} + n^{1-4r+\nu} \omega_{r-\nu}(\varphi) (f^{(i)},2) \right) \right\}
\]

\[
\leq c \left\{ \omega_{r-\nu}(\varphi) \left( f^{(i)}(\cdot) \frac{1}{n} + n^{1-4r+\nu} (2n)^{-\nu} \omega_{r-\nu}(\varphi) \left( f^{(i)}(\cdot) \frac{1}{n} \right) \right) \right\} \leq c \omega_{r-\nu}(\varphi) \left( f^{(i)}(\cdot) \frac{1}{n} \right)
\]

Proof of Theorem 1.1. Suppose that \( \nu = 0,1,...,r \) for \( x \in [-1,1] \). By the Markov's inequality, we have

\[
g \left( \cdot \right) - L_n \left( g \left( \cdot \right) \right) \leq c \left( \frac{1}{n} \right)^{\nu} \|g^{(r)}\|_{(\varphi)} \left( \frac{1}{n} \right), \quad g \in L_{(\varphi)}^{r,r+\nu} \tag{3.19}
\]

\[
L_n \left( g \left( \cdot \right) \right) \leq c, \quad n = 1,2,.... \tag{3.20}
\]

Hence, for any \( f \in L_{(\varphi)}^{r,r+\nu} \), by the corollary of Theorem 2.3, it follows that

\[
\|f \left( \cdot \right) - L_n \left( f \left( \cdot \right) \right)\|_{(\varphi)} \leq \omega_{r-\nu}(\varphi) \left( f^{(i)}(\cdot) \frac{1}{n} \right).
\]

Therefore, we can obtain

\[
\|f^{(i)}(\cdot) - F_n^{(i)} f \|_{(\varphi)} \leq c \omega_{r-\nu}(\varphi) \left( f^{(i)}(\cdot) \frac{1}{n} \right).
\]

By similarly reasons as in (3.8) and (3.9), from (3.16) and (3.17), we easily prove

\[
\int_{-\pi}^{\pi} |f(\cdot) - L_n(\cdot)| dt \leq c n^{-\nu}, \quad i = 0,1, ..., r. \tag{3.18}
\]

Let (Garidi, 1991)

\[
L_n(f,x) := \int_{-\pi}^{\pi} \{ f(\cdot) + (-1)^{i+1} \Delta_i^{(i)} (f,x) \} \tilde{K}_n(t) dt \in T_n. \tag{3.19}
\]
4. Conclusion

In conclusion, in this paper we gave definition of another class of functions (Chen, 1964) that is wider than the classical Orlicz spaces \( L^\varphi \). The wider class denoted by \( L^{\varphi^*} \) and the generating function \( \varphi \) of \( L^{\varphi^*} \) is not necessary to be convex (Chen, 1964). Then we proved some theorems on simultaneous approximation by trigonometric or algebraic polynomials in Orlicz spaces constructed by Young functions belonging to a reasonably wide class.

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تعريف آني بواسطة حدوديات على فضاءات أورلزن مؤكدة بواسطة دوال يونغ شبة المحدبة

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خلاصة

تقوم في هذا البحث بإثبات بعض النظريات على تقريب آني بواسطة حدوديات جبرية أو مثلثية في فضاءات أورلزن ومشتقة بواسطة دوال يونغ المنتج إلى صنف كبير بشكل معقول.