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Norms and compactness of operators on absolute weighted mean summable series

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Abstract

In a recent paper we characterized the classes of triangular matrix transformations mapping from the spaces $\left|\bar{N}_p\right|$ and $\left|\bar{N}_p^{\theta}\right|_k$ into the spaces $\left|\bar{N}_q^{\theta}\right|_k$ and $\left|\bar{N}_q\right|$, respectively, where the spaces $\left|\bar{N}_p^{\theta}\right|_k$, k≥1, series summable by absolute summability method. In the present paper we show that each element of these classes corresponds to a bounded linear operator, and determine exactly or estimate their norms and those in some well known classes. Also, we characterize compact operators in these classes by using Hausdorff measure of noncompactness.

Keywords: Absolute weighted summability; bounded operator; compact operator; Hausdorff measure of non compactness; matrixtrans formations.

Mathematics subject classifications (2010): 40D09, 40F05, 46B45, 46B50

1. Introduction

Let $\sum a_\nu$ be a given infinite series with s_n as its nth partial sums. Let $A=(a_{n\nu})$ be an arbitrary infinite matrix of complex numbers and (θ_n) be a positive sequence. By $A(s)=(A_n(s))$ we denote the A-transform of the sequence $s=(s_n)$, i.e.,

$$A_n(s)=\sum_{\nu=0}^{\infty} a_{n\nu}\, s_\nu \tag{1}$$

provided that the series are convergent for $=0,1,\dots$. A series $\sum a_\nu$ is said to be summable $|A,\theta|_k$, $k\geq 1$, if

$$\sum_{n=1}^{\infty}\theta_n^{k-1}\left|A_n(s)-A_{n-1}(s)\right|^k<\infty\,,$$

(Sarıgöl, 2010). If $A=(\bar{N},p_n)$ and $\theta_n=P_n/p_n$, then the summability $|A,\theta|_k$ is reduced to the summabilities $|\bar{N},p_n,q_n|_k$ and $|\bar{N},p_n|_k$, (Sulaiman, 1992; Bor & Thorpe, 1987), respectively. Also, $|A,\theta|_k=|C,\alpha|_k$ for $A=(C,\alpha)$ and $\theta_n=n$, (Flett, 1957). By a weighted mean matrix we mean one such that

$$a_{n\nu}=\begin{cases} p_\nu/P_n\,, & 0\leq\nu\leq n\\ 0, & \nu>n\end{cases}$$

where

$$P_n=p_0+p_1+\cdots+p_n\to\infty \text{ as } n\to\infty,\qquad P_{-1}=p_{-1}=0\,. \tag{2}$$

By $\left|\bar{N}_p^{\theta}\right|_k$, we donete the set of series summable by the summability method $|\bar{N},p_n,\theta_n|_k$. Then it can be easily seen that

$$\left|\bar{N}_p^{\theta}\right|_k=\left\{a=(a_n):\ \sum_{n=1}^{\infty}\theta_n^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n}P_{\nu-1}a_\nu\right|^k<\infty\right\},k\geq 1,$$

and so it means that a series $\sum a_\nu$ is summable $|\bar{N},p_n,\theta_n|_k$ if and only if the sequence $a=(a_\nu)\in\left|\bar{N}_p^{\theta}\right|_k$. Also, it is routine to verify that $\left|\bar{N}_p^{\theta}\right|_k$ is a Banach space with respect to the norm

$$\|a\|_{|\bar{N}_p^{\theta}|_k}=\left\{|a_0|^k+\sum_{n=1}^{\infty}\theta_n^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n}P_{\nu-1}a_\nu\right|^k<\infty\right\}^{1/k},k\geq 1, \tag{3}$$

with $\theta_n=1$ (Sarıgöl, 1991). In addition, the space $\left|\bar{N}_p\right|(M,r,q,s)$ generalizing the space $\left|N_p^{\theta}\right|_k$ with $\theta_n=1$ was studied by Altin *et al.*(2004).

Let X ve Y be two sequence spaces. We say that the matrix A defines a matrix transformation from X into Y, and denote it by writing $A: X \to Y$ if $A(s)$ exists and is in Y whenever $s \in X$. By (X, Y), we denote the class of all infinite matrices which map X into Y. For a long time, problems of comparision of summability methods and summability factors have widely been examined by many authors (Bor & Thorpe, 1987; Bosanquet, 1950; Kuttner, 1985; Mazhar, 1971; Mehdi, 1960; McFadden, 1942; Mohapatra & Das, 1975; Orhan & Sarıgöl, 1993; Sarıgöl, 1991; Sarıgöl, 1993; Sarıgöl, 2011; Sarıgöl & Bor, 1995; Sarıgöl, 2010; Sarıgöl, 2015; Sulaiman, 1992; Sunouchi, 1949). Now, according to other viewpoint we note that most of these results correspond to the special matrix transformations $I, W \in (X, Y)$, where I is identity matrix and W is the matrix defined by $w_{nv} = \varepsilon_n$ for $v = n$, zero otherwise, respectively. In a recent paper (Sarigol, 2011), in this way, the following classes of triangular matrix transformations in $\left(|\bar{N}_p|, |\bar{N}_q^\theta|_k\right)$ and $\left(|\bar{N}_p^\theta|_k, |\bar{N}_q|\right)$ have been characterized, which also include some well known results of Bosanquet (1950), Orhan & Sarıgöl (1993), and Sunouchi (1949).

Theorem 1.1. Assume that $A = (a_{nv})$ is a triangular matrix of complex numbers and (θ_n) is a positive sequence. Then $A \in \left(|\bar{N}_p|, |\bar{N}_q^\theta|_k\right)$, $1 \le k < \infty$, if and only if

$$a_{vv} = O\left\{\theta_v^{-\frac{1}{k^*}} \frac{p_v Q_v}{P_v q_v}\right\},$$

$$\sum_{n=v+1}^{\infty} \left| \sum_{m=v}^{n} \frac{\theta_n^{\frac{1}{k^*}} q_n Q_{m-1}}{Q_n Q_{n-1}} (a_{mv} - a_{m,v+1}) \right|^k = O\left\{\left(\frac{p_v}{P_v}\right)^k\right\},$$

$$\sum_{n=v+1}^{\infty} \left| \sum_{m=v+1}^{n} \frac{\theta_n^{\frac{1}{k^*}} q_n Q_{m-1}}{Q_n Q_{n-1}} a_{m,v+1} \right|^k = O(1), as\ v \to \infty.$$

Theorem 1.2. Let $1 < k < \infty, 1/k + 1/k^* = 1$. Assume that $A = (a_{nv})$ is a triangular matrix of complex numbers and (θ_n) is a positive sequence. Then $A \in \left(|\bar{N}_p^\theta|_k, |\bar{N}_q|\right)$ if and only if

$$\sum_{v=1}^{\infty} \frac{1}{\theta_v} \left(\sum_{n=v}^{\infty} \left| \sum_{m=v}^{n} \frac{q_n Q_{m-1}}{p_v Q_n Q_{n-1}} (P_v a_{mv} - P_{v-1} a_{m,v+1}) \right| \right)^{k^*} < \infty.$$

We also need the following results for our investigations:

Lemma 1.3. (Stieglitz & Tietz, 1977). Let $1 < k < \infty$. Then, $A \in (l_k, l)$ if and only if

$$\|A\|_{(l_k, l)} = \sup_N \left\{ \sum_{v=0}^{\infty} \left| \sum_{n \in N} a_{nv} \right|^{k^*} \right\}^{\frac{1}{k^*}} < \infty,$$

where N is any finite set of positive numbers.

It may be noted that the norm $\|A\|_{(l_k, l)}$ is exactly determined by Lemma 1.3. However, it exposes a rather difficult condition to apply in applications. So the following lemma, which gives equivalent norm, is more useful in many cases.

Lemma 1.4. (Sarıgöl, 2015). Let $1 < k < \infty$. Then, $A \in (l_k, l)$ if and only if

$$\|A\|' = \left\{ \sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} \right\}^{\frac{1}{k^*}} < \infty,$$

and there exists $1 \le \xi \le 4$ such that $\|A\|'_{(l_k, l)} = \xi \|A\|_{(l_k, l)}$.

The second part of this Lemma is easily seen by following the lines in Sarıgöl (2015) that

$$\|A\|_{(l_k, l)} \le \|A\|'_{(l_k, l)} \le 4\|A\|_{(l_k, l)},$$

From this inequality we can obtain the required result.

Lemma 1.5. (Maddox, 1970). Let $1 \le k < \infty$. Then, $A \in (l, l_k)$ if and only if

$$\|A\|_{(l, l_k)} = \sup_v \left\{ \sum_{n=0}^{\infty} |a_{nv}|^k \right\}^{\frac{1}{k}} < \infty.$$

Lemma 1.6. Let $1 \le k < \infty$. Then, the spaces $|\bar{N}_p^\theta|_k$ and l_k are isometrically isomorphic.

Proof. Let us consider the mapping $T: |\bar{N}_p^\theta|_k \to l_k$ defined by

$$T_n(a) = \frac{\theta_n^{\frac{1}{k^*}} p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, T_0(a) = a_0. \quad (4)$$

Then, it is clear that T is linear and one to one. Now, given $y \in l_k$ for surjectivity. Take, for $n \ge 1$,

$$a_n = \theta_n^{-\frac{1}{k^*}} \frac{P_n}{p_n} y_n - \theta_{n-1}^{-\frac{1}{k^*}} \frac{P_{n-2}}{p_{n-1}} y_{n-1}, a_0 = y_0. \quad (5)$$

Then, $T(a) = y$ and so $\in |\bar{N}_p^\theta|_k$, which implies that T is surjective. Also, it preserves the norm, i.e., $\|a\|_{|\bar{N}_p^\theta|_k} = \|T(a)\|_{l_k}$.

Lemma 1.7. Let $1 \le k < \infty$. If (p_n) is a sequence of positive numbers satisfying

$P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$, $\quad P_{-1} = p_{-1} = 0.$

then

$$\frac{1}{k} \leq P_{\nu-1} \sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \leq 1 \,. \tag{6}$$

Proof. Let us consider the function $f : [0,1) \to \mathbb{R}$ defined by

$$f(x) = \frac{1-x}{1-x^k} \,.$$

Then, since f is a positive decreasing function and $f(x) \to \frac{1}{k}$ as $x \to 1-0$, we have $\frac{1}{k} \leq f(x) \leq 1$ for all $x \in [0,1)$. So, if we take $x = \frac{P_{n-1}}{P_n}$, it follows that

$$\frac{1}{k}\left(\frac{1}{P_{n-1}^k} - \frac{1}{P_n^k}\right) \leq \frac{p_n}{P_n P_{n-1}^k} \leq \frac{1}{P_{n-1}^k} - \frac{1}{P_n^k} \,,$$

which implies (6).

2. The hausdorff measure of noncompactness

If R and H are subsets of a metric space (X,d) and $\varepsilon > 0$ then S is called an ε-net of H, if, for every $h \in H$, there exists an $s \in S$ such that $d(h,s) < \varepsilon$; if S is finite, then the ε-net S of H is called a *finite ε-net* of H. Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is called compact its domain is all of X and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. We denote the class of such operators by $C(X,Y)$. If Q is a bounded subset of the metric space X, then the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q) = \{\, \varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X \,\},$$

and χ is called the Hausdorff measure of noncompactness.

Lemma 2.1. (Rakočević, 1998). Let Q be a bounded subset of the normed space X where $X = l_k$, for $1 \leq k < \infty$ or $X = c_0$. If $P_n : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, \dots, x_r, 0, \dots)$ for all $x \in X$, then

$$\chi(Q) = \lim_{r \to \infty} \left(\sup_{x \in Q} \|(I - P_r)(x)\| \right).$$

Let X and Y be Banach spaces and χ_1 and χ_2 be Hausdorff measures on X and Y, then, the linear operator $L : X \to Y$ is said to be (χ_1, χ_2) –bounded if $L(Q)$ is a bounded subset of Y and there exsists a positive constant M such that $\chi_2(L(Q)) \leq M\,\chi_1(Q)$ for every bounded subset Q of X. If an operator L is (χ_1, χ_2) –bounded then the number

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{\, M > 0 : \chi_2(L(Q)) \leq M\,\chi_1(Q) \text{ for all bounded } Q \subset X \,\}$$

is called the (χ_1,χ_2)-measure noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_{(\chi,\chi)} = \|L\|_{\chi}$.

Lemma 2.2. (Malkowsky & Rakočević, 2000). Let X and Y be Banach spaces, $L \in B(X,Y)$ and $S_X = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere in X. Then

$$\|L\|_{\chi} = \chi(L(S_X)).$$

Furthermore, $L \in C(X,Y)$ if and only if $\|L\|_{\chi} = 0$, and the Hausdorff measure of noncompactness satisfy the inequality $\|L\|_{\chi} \leq \|L\|$ (Malkowsky & Rakočević, 2000).

Let $X_T = \{x \in w : T(x) \in X\}$, where $T = (t_{n\nu})$ is a triangular infinite matrix. Then, we have

Lemma 2.3. (Malkowsky & Rakočević, 2007). Let X be normed sequence space and χ_T and χ denote the Hausdorff measures of noncompactness on M_{X_T} and M_X, the collections of all bounded sets in X_T and X, respectively. Then, $\chi_T(Q) = \chi(T(Q))$ for all $Q \in M_{X_T}$.

3. Main results

In the present paper we show that each element of the classes $\left(|\bar{N}_p|, |\bar{N}_q^{\theta}|_k\right)$ and $\left(|\bar{N}_p^{\theta}|_k, |\bar{N}_q|\right)$ corresponds to a bounded linear operator, and determine exactly or estimate their norms and also those in some well- known classes. Besides, we characterize compact operators in these classes by using Hausdorff measure of noncompactness.

Theorem 3.1. Let $A = (a_{n\nu})$ be a triangular infinite matrix and (θ_n) be a positive sequence. Then $\left(|\bar{N}_p|, |\bar{N}_q^{\theta}|_k\right) \subset B\left(|\bar{N}_p|, |\bar{N}_q^{\theta}|_k\right)$, $1 \leq k < \infty$, i.e., every matrix $A \in \left(|\bar{N}_p|, |\bar{N}_q^{\theta}|_k\right)$ defines an operator $L_A \in B\left(|\bar{N}_p|, |\bar{N}_q^{\theta}|_k\right)$ such that $L_A(x) = A(x)$ for all $x \in |\bar{N}_p|$, and if $A \in \left(|\bar{N}_p|, |\bar{N}_q^{\theta}|_k\right)$, then

$$\|A\|_{\left(|\bar{N}_p|, |\bar{N}_q^{\theta}|_k\right)} = \sup_{\nu} \left\{ \sum_{n=\nu}^{\infty} |d_{n\nu}|^k \right\}^{\frac{1}{k}}$$

and

$$\|A\|_{\chi} = \lim_{r \to \infty} \sup_{\nu} \left\{ \sum_{n=r+1}^{\infty} |d_{n\nu}|^k \right\}^{\frac{1}{k}},$$

where the matrix $D = (d_{n\nu})$ defined by

$$d_{nv}=\begin{cases} a_{00}, & n=0, v=0 \\ \dfrac{\theta_n^{\frac{1}{k^*}} q_n}{Q_n Q_{n-1}} \displaystyle\sum_{m=v}^{n} \frac{Q_{m-1}}{p_v}\left(P_v a_{mv}-P_{v-1} a_{m,v+1}\right), & 0 \leq v \leq n \end{cases} \tag{7}$$

Proof. Since $\left|\bar{N}_p^\theta\right|_k$ is a Banach space for $k \geq 1$, it is sufficient to prove the first part of the Theorem that coordinate functional $P_n:\left|\bar{N}_p^\theta\right|_k \longrightarrow \mathbb{C}$ is bounded. In fact, by (3) and (4), we have

$$\|a\|_{\left|\bar{N}_p^\theta\right|_k}=\left\{\left|T_0(a)\right|^k+\sum_{n=1}^{\infty}\left|T_n(a)\right|^k\right\}^{1/k} \tag{8}$$

and so it follows from the inversion of (4) and (8) that

$$\|P_n(a)\| \leq\left(\theta_n^{-\frac{1}{k^*}} \frac{P_n}{p_n}+\theta_{n-1}^{-\frac{1}{k^*}} \frac{P_{n-2}}{p_{n-1}}\right)\|a\|_{\left|\bar{N}_p^\theta\right|_k}.$$

Now, by following the lines in Sarıgöl (2011), we get for n ≥ 1,

$$A_0(a)=a_{00} T_0(a),$$

$$A_n(a)=\sum_{v=0}^{n}\left\{\theta_n^{\frac{1}{k^*}} \frac{q_n}{Q_n Q_{n-1}} \sum_{m=v}^{n} \frac{Q_{m-1}}{p_v}\left(P_v a_{mv}-P_{v-1} a_{m,v+1}\right)\right\} T_v(a)=\sum_{v=0}^{\infty} d_{nv} T_v(a)$$

This gives that $A \in\left(\left|\bar{N}_p\right|,\left|\bar{N}_q^\theta\right|_k\right) \Leftrightarrow D \in\left(l, l_k\right)$. On the other hand, we get $A=L^{-1} \circ D \circ T$ if the maps $T:\left|\bar{N}_p\right| \longrightarrow l$ and $L:\left|\bar{N}_q^\theta\right|_k \longrightarrow l_k$ are defined by

$$T_n(a)=\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, T_0(a)=a_0,$$

and

$$L_n(a)=\frac{\theta_n^{\frac{1}{k^*}} q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v, T_0(a)=a_0,$$

respectively. Therefore, by Lemma 1.6, we obtain

$$\begin{aligned} \|A\|_{\left(\left|\bar{N}_p\right|,\left|\bar{N}_q^\theta\right|_k\right)} &=\sup_{a \neq \theta} \frac{\left\|L^{-1}(D(T(a))\right\|_{l_k}}{\|a\|_{\left|\bar{N}_p\right|}} \\ &=\sup_{a \neq \theta} \frac{\|D(T(a)\|_{l_k}}{\|a\|_{\left|\bar{N}_p\right|}} \\ &=\|D\|_{\left(l, l_k\right)} \end{aligned}$$

which completes the proof of the second part together with applying the matrix D to Lemma 1.5.

Finally, let $S=\left\{x \epsilon\left|\bar{N}_q\right|:\|x\| \leq 1\right\}$. Then, it follows from Lemma 2.1, Lemma 2.2 and Lemma 2.3 that

$$\begin{aligned} \|A\|_\chi &=\chi(AS)=\chi(LAS)=\chi\left(LAT^{-1} S\right)=\chi(DS) \\ &=\lim_{r \rightarrow \infty} \sup_{y \in TS}\left\|\left(I-P_r\right) D(y)\right\|, \end{aligned}$$

where $P_r: l_k \longrightarrow l_k$ $(r=0,1, \ldots)$ is defined by $P_r(x)=\left(x_0, x_1, \ldots x_r, 0, \ldots\right)$. Therefore, if we define the matrix $D^{(r)}=\left(\bar{d}_{nv}\right)$ by

$$\bar{d}_{nv}=\begin{cases} 0, & 0 \leq n \leq r \\ d_{nv}, & n>r \end{cases},$$

then

$$\sup_{y \in TS}\left\|\left(I-P_r\right) D(y)\right\|=\left\|D^{(r)}\right\|_{\left(l, l_k\right)},$$

which completes the proof by Lemma 1.5.

By Lemma 2.2 and Theorem 3.1, we directly obtain the following result which characterize the compact operators in the class $\left(\left|\bar{N}_p\right|,\left|\bar{N}_q^\theta\right|_k\right)$.

Corollary 3.2. Under conditions of Theorem 3.1, $A \in\left(\left|\bar{N}_P\right|,\left|\bar{N}_q^\theta\right|_k\right)$ is compact if and only if

$$\lim_{r \rightarrow \infty} \sup_v\left(\sum_{n=r+1}^{\infty}\left|d_{nv}\right|^k\right)^{\frac{1}{k}}=0.$$

Applying Theorem 3.1 to some special cases, we determine exactly or estimate the norms of bounded linear operators in well known matrix classes. First, we determine exactly the norm of the matrix transformation characterized by Bosanquet (1950) and Sunouchi, (1949).

Corollary 3.3. If $I \in\left(\left|\bar{N}_p\right|,\left|\bar{N}_q\right|\right)$, i.e., $\left|\bar{N}_p\right| \subset\left|\bar{N}_q\right|$, then

$$\|I\|_{\left(\left|\bar{N}_p\right|,\left|\bar{N}_q\right|\right)}=\sup_v\left\{\frac{q_v P_v}{Q_v p_v}+\left|1-\frac{q_v P_v}{Q_v p_v}\right|\right\}.$$

Take $\theta_n=1$ for all $n \in \mathbb{N}$, $k=1$ and $A=I$ in Theorem 3.1.

Corollary 3.4. If $I \in\left(\left|R_p\right|,\left|R_q\right|_k\right)$ for $k \geq 1$, then

$$\|I\|_{\left(|R_p|,|R_q|_k\right)} = \sup_{\nu}\left\{\left(\nu^{\frac{1}{k^*}}\frac{q_\nu P_\nu}{Q_\nu p_\nu}\right)^k + \left|Q_\nu - \frac{q_\nu P_\nu}{p_\nu}\right|^k \sum_{n=\nu+1}^{\infty}\left(\frac{n^{\frac{1}{k^*}}q_n}{Q_n Q_{n-1}}\right)^k\right\}^{\frac{1}{k}}.$$

Take $\theta_n = n$ for all $n \in \mathbb{N}$ and $A = I$ in Theorem 3.1. This matrix transformation was characterized by Orhan & Sarıgöl (1993).

Corollary 3.5. If $I \in \left(|\bar{N}_p|, |\bar{N}_q|_k\right)$ for $k \geq 1$, then there exists $\frac{1}{k} \leq \xi \leq 1$ such that

$$\|I\|_{\left(|\bar{N}_p|,|\bar{N}_q|_k\right)} = \sup_{\nu}\left\{\frac{q_\nu}{Q_\nu}\left(\frac{P_\nu}{p_\nu}\right)^k + \xi\left|1 - \frac{q_\nu P_\nu}{Q_\nu p_\nu}\right|^k\right\}^{\frac{1}{k}}.$$

On considering $\theta_n = Q_n/q_n$ for all $n \in \mathbb{N}$ and $A = I$ in Theorem 3.1, also using Lemma 1.7, we get the matrix transformation studied in Sarıgöl (2011).

Theorem 3.6. Let $1 < k < \infty, \frac{1}{k} + \frac{1}{k^*} = 1$, $A = (a_{n\nu})$ be a triangular infinite matrix and (θ_n) be a positive sequence. Then, $\left(|\bar{N}_p^\theta|_k, |\bar{N}_q|\right) \subset B\left(|\bar{N}_p^\theta|_k, |\bar{N}_q|\right)$, i.e., every matrix $A \in \left(|\bar{N}_p^\theta|_k, |\bar{N}_q|\right)$ defines an operator $L_A \in B\left(|\bar{N}_p^\theta|_k, |\bar{N}_q|\right)$ such that $L_A(x) = A(x)$ for all $x \in |\bar{N}_p^\theta|_k$, and if $A \in \left(|\bar{N}_p^\theta|_k, |\bar{N}_q|\right)$, then there exist $1 \leq \xi \leq 4$ such that

$$\|A\|_{\left(|\bar{N}_p^\theta|_k,|\bar{N}_q|\right)} = \frac{1}{\xi}\left\{\sum_{\nu=0}^{\infty}\left(\sum_{n=\nu}^{\infty}|\hat{d}_{n\nu}|\right)^{k^*}\right\}^{\frac{1}{k^*}}$$

and

$$\|A\|_\chi = \frac{1}{\xi}\lim_{r\to\infty}\left\{\sum_{\nu=0}^{\infty}\left(\sum_{n=r+1}^{\infty}|\hat{d}_{n\nu}|\right)^{k^*}\right\}^{\frac{1}{k^*}},$$

where the matrix $\hat{D} = (\hat{d}_{n\nu})$ is defined by

$$\hat{d}_{n\nu} = \begin{cases} a_{00}, & \nu = 0, n = 0 \\ \dfrac{q_n}{Q_n Q_{n-1}}\displaystyle\sum_{m=\nu}^{n}\frac{\theta_\nu^{-\frac{1}{k^*}}Q_{m-1}}{p_\nu}\left(P_\nu a_{m\nu} - P_{\nu-1}a_{m,\nu+1}\right), & 0 \leq \nu \leq n \\ 0, & \nu > n \end{cases} \tag{9}$$

Proof. The first part is clear. Now, as in Sarigol (2011), we get, for n ≥ 1,

$$A_0(a) = a_{00}\hat{T}_0(a),$$

for n ≥ 1,

$$A_n(a) = \sum_{\nu=0}^{n}\left\{\frac{q_n}{Q_n Q_{n-1}}\sum_{m=\nu}^{n}\frac{\theta_\nu^{-\frac{1}{k^*}}Q_{m-1}}{p_\nu}\left(P_\nu a_{m\nu} - P_{\nu-1}a_{m,\nu+1}\right)\right\}\hat{T}_\nu(a)$$

$$= \sum_{\nu=0}^{\infty}\hat{d}_{n\nu}\,\hat{T}_\nu(a).$$

This gives us that $A \in \left(|\bar{N}_p^\theta|_k, |\bar{N}_q|\right) \Leftrightarrow \hat{D} \in (l_k, l)$. Also $A = \left(\hat{L}\right)^{-1} \circ \hat{D} \circ \hat{T}$, where $\hat{T} : |\bar{N}_p^\theta|_k \longrightarrow l_k$ and $\hat{L} : |\bar{N}_q| \longrightarrow l$ are defined by

$$\hat{T}_n(a) = \frac{\theta_n^{\frac{1}{k^*}}p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n}P_{\nu-1}a_\nu\,, \hat{T}_0(a) = a_0,$$

and

$$\hat{L}_n(a) = \frac{q_n}{Q_n Q_{n-1}}\sum_{\nu=1}^{n}Q_{\nu-1}a_\nu\,, \hat{L}_0(a) = a_0,$$

respectively. So, it follows from Lemma 1.6 that

$$\|A\|_{\left(|\bar{N}_p^\theta|_k,|\bar{N}_q|\right)} = \|\hat{D}\|_{(l_k,l)}$$

which completes the proof by applying the matrix $\hat{D}$ to Lemma 1.4. The last part is similar to the above, so the proof is omitted.

Corollary 3.7. Under conditions of Theorem 3.6, $A \in \left(|\bar{N}_p^\theta|_k, |\bar{N}_q|\right)$ is compact if and only if

$$\lim_{r\to\infty}\sum_{\nu=0}^{\infty}\left(\sum_{n=r+1}^{\infty}\left|\hat{d}_{n\nu}\right|\right)^{k^*}=0.$$

Corollary 3.8. Let $k>1$. If $I\in\left(\left|\bar{N}_p^\theta\right|_k,\left|\bar{N}_q\right|\right)$, then

$$\|I\|_{\left(\left|\bar{N}_p^\theta\right|_k,\left|\bar{N}_q\right|\right)}=\left\{\sum_{\nu=0}^{\infty}\frac{1}{\theta_\nu}\left(\frac{q_\nu P_\nu}{Q_\nu p_\nu}+\left|1-\frac{q_\nu P_\nu}{Q_\nu p_\nu}\right|\right)^{k^*}\right\}^{\frac{1}{k^*}}$$

On considering $A=I$ in Theorem 3.6, we get the matrix transformation studied in Sarıgöl (1993).

Now, if we put $\theta_0=1$ and $\theta_n=n$ for all $n\in\mathbb{N}$ and $A=I$ in Corollary 3.8, then it follows that

$$\|I\|_{\left(\left|\bar{N}_p^\theta\right|_k,\left|\bar{N}_q\right|\right)}=\infty,$$

which gives a result in Sarıgöl (1993).

Corollary 3.9. If $k>1$, then $I\notin\left(\left|\bar{N}_p\right|_k,\left|\bar{N}_q\right|\right)$ for all positive sequences (p_n) and (q_n) satisfying (1.2), i.e., there exist a series which is summable by summable $|\bar{N},p_n|_k$ but not summable $|\bar{N},q_n|$.

4. Conclusion

In the present paper, showing that any triangular matrix transformation mapping from the spaces $\left|\bar{N}_p\right|$ and $\left|\bar{N}_p^\theta\right|_k$ into the spaces $\left|\bar{N}_q^\theta\right|_k$ and $\left|\bar{N}_q\right|$, respectively, corresponds to a bounded linear operator, we determine or estimate its norm and also give necessary and sufficient conditions for it to be compact by means of Hausdorff measure of noncompactness, where the spaces $\left|\bar{N}_p^\theta\right|_k$, k≥1, series summable by absolute summability method. And so it has been brought a different perspective and studying field.

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معايير و تراص مؤثرات على متسلسلة قابلة للجمع و ذات وسط مطلق موُزن

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خلاصة

لقد قمنا، في بحث سابق، بإيجاد خصائص أصناف تحويلات مصفوفيه مثلثية بين فضاءات متسلسلات قابلة للجمع و ذلك بواسطة طريقة الجمع المطلق. و نقوم في هذا البحث بإثبات أن كل صنف من هذه الاصناف يقابله مؤثر خطي محدود، كما نقوم بإيجاد أو تقدير، معيار كل صنف من الأصناف المعروفة جيداً. كما نحدد خصائص المؤثرات المتراصة لهذه الأصناف و ذلك باستخدام قياس هاوسدورف لعدم التراص.