Kuwait J. Sci. **40** (2) pp. 1-16, 2013

Some results on Laplace-Stieltjes transform

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ABSTRACT

Two general theorems on Laplace-Stieltjes transform are established. One theorem is a generalized convolution theorem, while the second is a generalized Parseval theorem for a operator with kernel k(s,t). Several special cases can be derived from these theorems. One section is devoted to applications of these theorems to evaluate some integrals.

Keywords: Laplace-Stieltjes; transform; convolution; integrals.

INTRODUCTION

The ordinary Laplace transform and its applications to solve a variety of problems in engineering, physics and applied mathematics are well known. Laplace-Stieltjes transform and its properties are very useful in applied probability, queueing theory (Feller, 1971) and some other problems (Widder, 1941; Ridout, 2009). It turns out that the Laplace-Stieltjes transform of the distribution function of an arbitrary non-negative random variable shares many of the properties of the probability generating function of an arbitrary non-negative integer-valued random variable, and this fact accounts for its usefulness in queueing theory.

Convolution theorem for Laplace transform, Efros theorem, Parseval theorems and their generalizations are well known in the literature due to their various applications in analysis and boundary value problems (Ben-Nakhi & Kalla, 2003; Galué *et al.*, 2007; Kalla, 1970). Here, first we establish a general convolution type theorem for Laplace-Stieltjes transform. The second theorem is for a general integral operator whose kernel is k(t, x). As an application of these theorems, some integrals are evaluated, which may be useful in boundary value problems involving fractional differential equations.

The results derived here may be useful in solving problems of temperature fields in oil strata (Boyadjiev *et al.*, 2005), fractional kinetic equations (Saxena *et al.*, 2008), fractional Schrödinger equation of quantum mechanics (Saxena *et al.*, 2010) and similar related problems.

MAIN RESULTS

In this section, we establish two theorems. The first result is a generalized convolution theorem for Laplace-Stieltjes transform, while the second may be considered as a general Parseval theorem.

We denote by

$$(L_S f)(p) = \int_0^\infty e^{-pt}\, df(t)$$

the Laplace-Stieltjes transform of $f \in BV_{loc}(R_+)$ and L_S^{-1} stands for the inverse transform.

Theorem 1. *Let g be a Laplace-Stieltjes transformable function and let* $W(t,\tau)$ *be a two parameters function such that* $(L_S g)(.)W(.,\tau)$ *is in the range of the Laplace-Stieltjes transform for every* $\tau \geq 0$. *Let* $k(t,\tau) := L_S^{-1}[(L_S g)(.)W(.,\tau)](t)$. *Assume that* $f \in BV_{loc}(R_+)$ *is such that* $C(t) := \int_0^\infty k(t,\tau)df(\tau)$ *and* $\int_0^\infty W(p,\tau)df(\tau)$ *both exist for all* $t \geq 0$. *Then,*

$$\int_0^\infty e^{-pt}\, dC(t) = (L_S g)(p) \int_0^\infty W(p,\tau)df(\tau).$$

Proof. We first notice that

$$\begin{aligned} \textstyle\int_0^\infty e^{-pt}\, dC(t) &= \int_0^\infty e^{-pt}\, d_t\left(\int_0^\infty k(t,\tau)df(\tau)\right) \\ &= \int_0^\infty df(\tau) \int_0^\infty e^{-pt}\, d_t k(t,\tau) \end{aligned}$$

by an application of Fubini's Theorem (Arendt *et al.*, 2001; Widder, 1941). Hence we arrive at

$$\begin{aligned} \int_0^\infty e^{-pt}\, dC(t) &= \int_0^\infty df(\tau) \int_0^\infty e^{-pt} d_t k(t,\tau) \\ &= \int_0^\infty df(\tau)(L_S k(.,\tau))(p). \qquad (1) \end{aligned}$$

On the other hand by our assumption we have that $k(t, \tau) = L_S^{-1}[(L_S g)(.)W(., \tau)](t)$ .But then

$$(L_S k(., \tau))(p) = (L_S g)(p) W(p, \tau). \tag{2}$$

Now the claim follows from identities (1) and (2).

We have the following corollary:

Corollary. *If* $W(p, \tau) = e^{-p\tau}$ *then*

$$(L_S(g * f))(p) = (L_S g)(p)(L_S f)(p).$$

We next show a generalization of a theorem due to Kalla (1970). First we define the integral operator

$$Kf(x) = \int_0^\infty k(t, x) f(t) d\alpha(t),$$

in which $k(t, x) = g(x)G(t, x)$ where g is a positive function and G is a symmetric kernel, that is, $G(t, x) = G(x, t)$. Furthermore we assume that $k \in L^1(R_+ \times R_+)$.

Theorem 2. *Assume that f and h are such that* $x \to h(x) \int_0^\infty k(t, x) f(t) d\alpha(t)$, *and* $x \to \int_0^\infty k(t, x) f(t) \frac{d\alpha(t)}{g(t)}$ *are continuous and* $Kf \in L^1([0, \infty))$. *Then*

$$\int_0^\infty (Kh)(t) f(t) \frac{d\alpha(t)}{g(t)} = \int_0^\infty h(t)(Kf)(t) \frac{d\alpha(t)}{g(t)}. \tag{3}$$

Proof. Since

$$\begin{aligned} \int_0^\infty f(t) \int_0^\infty k(x, t) h(x) d\alpha(x) \frac{d\alpha(t)}{g(t)} &= \int_0^\infty h(x) \int_0^\infty k(x, t) f(t) \frac{d\alpha(t)}{g(t)} d\alpha(x) \\ &= \int_0^\infty h(x) \int_0^\infty g(t) G(t, x) f(t) \frac{d\alpha(t)}{g(t)} d\alpha(x) \\ &= \int_0^\infty h(x) \int_0^\infty k(t, x) f(t) d\alpha(t) \frac{d\alpha(x)}{g(x)}, \end{aligned}$$

the proof of Eq. (3) follows.

APPLICATIONS

Application of the theorem 1

In this section we consider some examples that prove the validity of the Theorem 1.

1) Let be $g(t) = \frac{t^{n+1}}{n+1}$, $k(t, \tau) = \sin(t\tau)$, $f(\tau) = \ln \tau$.

$$g(t) = \frac{t^{n+1}}{n+1} \Rightarrow (L_S g(t))(p) = \int_0^\infty e^{-pt}\, dg(t) = \int_0^\infty e^{-pt} t^n\, dt$$

from (Erdélyi, 1953)

$$\int_0^\infty e^{-pt} t^n\, dt = n!\, p^{-n-1}, \quad \Re(p) > 0,$$

therefore,

$$(L_S g(t))(p) = n!\, p^{-n-1}, \quad \Re(p) > 0. \tag{4}$$

On the other hand,

$$C(t) = \int_0^\infty \frac{\sin(t\tau)}{\tau}\, d\tau$$

exists because (Erdélyi, 1953)

$$\int_0^\infty \frac{\sin(xy)}{x}\, dx = \frac{\pi}{2}, \quad y > 0. \tag{5}$$

Since that,

$$k(t, \tau) = L_S^{-1}[(L_S g)(.)W(., \tau)](t)$$

then

$$(L_S k(., \tau))(p) = (L_S g)(p) W(p, \tau) \tag{6}$$

being,

$$(L_S k(., \tau))(p) = \int_0^\infty e^{-pt}\, d_t k(t, \tau).$$

So,

$$(L_S k(., \tau))(p) = \int_0^\infty e^{-pt}\, \tau \cos(t\tau)\, dt$$

$$= \frac{p\tau}{p^2 + \tau^2}, \qquad \Re(p) > 0, \tag{7}$$

where we use the following result (Erdélyi, 1953)

$$\int_0^\infty e^{-pt}\, \cos(at)\, dt = p\left(p^2 + a^2\right)^{-1}, \qquad \Re(p) > |Im\ a|.$$

From Eqs. (4), (6) and (7) we get

$$W(p, \tau) = \frac{p^{n+2}}{n!} \frac{\tau}{p^2 + \tau^2}, \qquad \Re(p) > 0,$$

and

$$\int_0^\infty W(p, \tau) df(\tau) = \frac{p^{n+2}}{n!} \int_0^\infty \frac{d\tau}{p^2 + \tau^2},$$

which using the known result (Erdélyi, 1953)

$$\int_0^\infty \frac{x^{s-1}}{x^2 + a^2} dx = \frac{\pi}{2} a^{s-2} \csc\left(\frac{\pi s}{2}\right), \qquad \Re(a) > 0, 0 < \Re(s) < 2, \tag{8}$$

is equivalent to

$$\int_0^\infty W(p, \tau) df(\tau) = \frac{\pi}{2} \frac{p^{n+1}}{n!}, \qquad \Re(p) > 0.$$

By a similar procedure we obtain the following results:

2) Let be $\quad g(t) = \frac{t^{n+1}}{n+1}, \qquad k(t, \tau) = \sin^2(t\tau), \qquad f(\tau) = \tau^{-1}.$

Then from Eq. (4)

$$(L_S g(t))(p) = n!\, p^{-n-1}, \qquad \Re(p) > 0.$$

$$(L_S k(., \tau))(p) = \int_0^\infty e^{-pt}\, \tau \sin(2t\tau)\, dt$$

$$= \frac{2\tau^2}{p^2 + (2\tau)^2}, \quad \Re(p) > 0,$$

since that (Erdélyi, 1953)

$$\int_0^\infty e^{-pt} \sin(at)\, dt = a\left(p^2 + a^2\right)^{-1}, \quad \Re(p) > |Im\ a|.$$

Therefore,

$$W(p, \tau) = \frac{p^{n+1}}{n!} \frac{2\tau^2}{p^2 + (2\tau)^2}, \quad \Re(p) > 0,$$

$$\int_0^\infty W(p, \tau) df(\tau) = -\frac{p^{n+1}}{n!} \int_0^\infty \frac{2d\tau}{p^2 + (2\tau)^2} = -\frac{\pi}{2} \frac{p^n}{n!}, \quad \Re(p) > 0$$

where we use again Eq. (8).

3) Let be $g(t) = \frac{t^{n+1}}{n+1}, \quad k(t, \tau) = 1 - \cos(t\tau), \quad f(\tau) = \tau^{-1}.$

Then from Eq. (4)

$$(L_S g(t))(p) = n!\, p^{-n-1}, \quad \Re(p) > 0.$$

$$(L_S k(., \tau))(p) = \int_0^\infty e^{-pt}\, \tau \sin(t\tau)\, dt$$

$$= \frac{\tau^2}{p^2 + \tau^2}, \quad \Re(p) > 0,$$

$$W(p, \tau) = \frac{p^{n+1}}{n!} \frac{\tau^2}{p^2 + \tau^2}, \quad \Re(p) > 0,$$

$$\int_0^\infty W(p, \tau) df(\tau) = -\frac{p^{n+1}}{n!} \int_0^\infty \frac{d\tau}{p^2 + \tau^2} = -\frac{\pi}{2} \frac{p^n}{n!}, \quad \Re(p) > 0.$$

4) Let be $g(t) = \frac{t^{n+1}}{n+1}, \quad k(t, \tau) = \frac{\cos(t\tau)}{2\tau}, \quad f(\tau) = \ln \tau.$

Then

$$(L_S g(t))(p) = n!\, p^{-n-1}, \quad \Re(p) > 0.$$

$$(L_S k(., \tau))(p) = -\frac{1}{2}\int_0^\infty e^{-pt} \sin(t\tau)\, dt$$

$$= -\frac{1}{2}\frac{\tau}{p^2+\tau^2}, \quad \Re(p) > 0,$$

$$W(p,\, \tau) = -\frac{p^{n+1}}{2n!}\frac{\tau}{p^2+\tau^2}, \quad \Re(p) > 0,$$

$$\int_0^\infty W(p,\tau)df(\tau) = -\frac{p^{n+1}}{2n!}\int_0^\infty \frac{d\tau}{p^2+\tau^2} = -\frac{\pi}{4}\frac{p^n}{n!}, \quad \Re(p) > 0$$

where we employ Eq. (8).

Application of the theorem 2

In this section we consider some examples that illustrate the use of the Theorem 2.

Let be

$$I_1 = \int_0^\infty f(t)\left(\int_0^\infty G(x,t)h(x)d\alpha(x)\right)d\alpha(t) \tag{9}$$

$$I_2 = \int_0^\infty h(x)\left(\int_0^\infty G(t,x)f(t)d\alpha(t)\right)d\alpha(x). \tag{10}$$

1) $h(x) = \sin mx, \quad G(t,\, x) = \frac{1}{t^2+x^2}, \quad f(t) = t^2 \sin bt, \quad \alpha(t) = \ln t.$

Then

$$I_1 = \int_0^\infty t^2 \sin bt\left(\int_0^\infty \frac{\sin mx}{t^2+x^2}\frac{dx}{x}\right)\frac{dt}{t}$$

$$I_2 = \int_0^\infty \sin mx\left(\int_0^\infty \frac{t \sin bt}{t^2+x^2}dt\right)\frac{dx}{x}. \tag{11}$$

The inner integral in I_1 can be evaluated by means of (Erdélyi, 1953)

$$\int_0^\infty \frac{\sin xy}{x(x^2+a^2)}dx = \frac{\pi}{2a^2}(1-e^{-ay}), \quad \Re(a) > 0, y > 0,$$

therefore

$$I_1 = \frac{\pi}{2}\left(\int_0^\infty \frac{\sin bt}{t}dt - \int_0^\infty \frac{e^{-mt}\sin bt}{t}dt\right)$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{b}{m}\right)\right] = \frac{\pi}{2}\tan^{-1}\left(\frac{m}{b}\right), \quad b, m > 0, \quad (12)$$

in view of Eq. (5) and (Erdélyi, 1953)

$$\int_0^\infty \frac{e^{-ax}\sin xy}{x}dx = \tan^{-1}\left(\frac{y}{a}\right), \quad \Re(a) > 0, \quad y > 0.$$

On the other hand, since that (Erdélyi, 1953)

$$\int_0^\infty \frac{x\sin xy}{x^2 + a^2}dx = \frac{\pi}{2}e^{-ay}, \quad \Re(a) > 0, \; y > 0, \quad (13)$$

we get from Eqs. (11), (12) and (13)

$$I_2 = \frac{\pi}{2}\int_0^\infty \frac{e^{-bx}\sin mx}{x}dx = \frac{\pi}{2}\tan^{-1}\left(\frac{m}{b}\right), \quad b, m > 0.$$

2) $h(x) = x^\mu$, $\quad G(t, x) = \frac{1}{(t^2+x^2)^{n+1}}$, $\quad f(t) = \frac{1}{(t^\nu+a)^{m+1}}$, $\quad \alpha(t) = \ln t$.

From Eqs. (9) and (10)

$$I_1 = \int_0^\infty \frac{1}{(t^\nu + a)^{m+1}}\left(\int_0^\infty \frac{x^{\mu-1}}{(t^2+x^2)^{n+1}}dx\right)dt \quad (14)$$

$$I_2 = \int_0^\infty x^{\mu-1}\left(\int_0^\infty \frac{1}{(t^2+x^2)^{n+1}}\frac{1}{(t^\nu + a)^{m+1}}dt\right)dx. \quad (15)$$

Applying in Eq. (14) the result (Gradshteyn & Ryzhik, 1965; Prudnikov *et al.*, 1992)

$$\int_0^\infty \frac{x^{\mu-1}d_x}{(qx^\nu + p)^{n+1}} = \frac{1}{\nu p^{n+1}}\left(\frac{p}{q}\right)^{\mu/\nu}\frac{\Gamma\left(\frac{\mu}{\nu}\right)\Gamma\left(1+n-\frac{\mu}{\nu}\right)}{\Gamma(n+1)}, \quad 0 < \frac{\mu}{\nu} < n+1, \quad (16)$$

we obtain

$$I_1 = \frac{1}{2}\frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(1+n-\frac{\mu}{2}\right)}{\Gamma(n+1)}\int_0^\infty \frac{t^{\mu-2n-2}}{(t^\nu+a)^{m+1}}\,dt, \quad 0<\mu<2n+2,$$

and using again Eq. (16)

$$I_1 = \frac{1}{2\nu}\frac{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(1+n-\frac{\mu}{2}\right)}{\Gamma(n+1)}\frac{\Gamma\left(\frac{\mu-2n-1}{\nu}\right)\Gamma\left(1+m-\left(\frac{\mu-2n-1}{\nu}\right)\right)}{\Gamma(m+1)}a^{(\mu-2n-1)/\nu-m-1},$$

$$0<\mu<2n+2,\ 0<\frac{\mu-2n-1}{\nu}<m+1,\ a\neq 0.$$

This result and the well known relation

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin \pi z}$$

lead us to the following expression

$$I_1 = \frac{1}{2\nu}\frac{(-1)^n\pi\,\Gamma\left(\frac{\mu}{2}\right)}{n!\sin\left(\frac{\mu}{2}\pi\right)\Gamma\left(\frac{\mu}{2}-n\right)}\,\frac{(-1)^m\pi\,\Gamma\left(\frac{\mu-2n-1}{\nu}\right)}{m!\sin\left(\left(\frac{\mu-2n-1}{\nu}\right)\pi\right)\Gamma\left(\left(\frac{\mu-2n-1}{\nu}\right)-m\right)}a^{(\mu-2n-1)/\nu-m-1}, \tag{17}$$

$$0<\mu<2n+2,\ 0<\frac{\mu-2n-1}{\nu}<m+1,\ a\neq 0,\ n,\ m=0,\ 1,\ 2,\cdots.$$

Then from Eqs. (15) and (17) we have

$$\int_0^\infty\int_0^\infty \frac{x^{\mu-1}}{(t^2+x^2)^{n+1}}\frac{1}{(t^\nu+a)^{m+1}}\,dt\,dx =$$

$$\frac{1}{2\nu}\frac{(-1)^n\pi\,\Gamma\left(\frac{\mu}{2}\right)}{n!\sin\left(\frac{\mu}{2}\pi\right)\Gamma\left(\frac{\mu}{2}-n\right)}\,\frac{(-1)^m\pi\,\Gamma\left(\frac{\mu-2n-1}{\nu}\right)}{m!\sin\left(\left(\frac{\mu-2n-1}{\nu}\right)\pi\right)\Gamma\left(\left(\frac{\mu-2n-1}{\nu}\right)-m\right)}a^{(\mu-2n-1)/\nu-m-1}, \tag{18}$$

$$0<\mu<2n+2,\ 0<\frac{\mu-2n-1}{\nu}<m+1,\ a\neq 0,\ n,\ m=0,\ 1,\ 2,\cdots.$$

As particular case from this result for $m=0$ we get

$$\int_0^\infty \int_0^\infty \frac{x^{\mu-1}}{(t^2+x^2)^{n+1}} \frac{1}{(t^\nu+a)} \; dt\, dx =$$

$$\frac{1}{2\nu} \frac{(-1)^n \pi\, \Gamma\left(\frac{\mu}{2}\right)}{n!\, \sin\left(\frac{\mu}{2}\pi\right) \Gamma\left(\frac{\mu}{2}-n\right)} \; \frac{\pi}{\sin\left(\left(\frac{\mu-2n-1}{\nu}\right)\pi\right)} a^{(\mu-2n-1)/\nu-1},$$

$$0<\mu<2n+2,\; 0<\frac{\mu-2n-1}{\nu}<1,\; a\neq 0,\; n=0,\,1,\,2,\cdots .$$

Now, if we put $\nu = 1$:

$$\int_0^\infty \int_0^\infty \frac{x^{\mu-1}}{(t^2+x^2)^{n+1}} \frac{1}{(t+a)} \; dt\, dx =$$

$$\frac{1}{2} \frac{(-1)^n \pi\, \Gamma\left(\frac{\mu}{2}\right)}{n!\, \sin\left(\frac{\mu}{2}\pi\right) \Gamma\left(\frac{\mu}{2}-n\right)} \; \frac{\pi}{\sin((\mu-2n-1)\,\pi)} a^{\mu-2n-2},$$

$$0<\mu<2n+2,\; a\neq 0,\; n=0,\,1,\,2,\cdots .$$

Which for $a = 1$reduces to

$$\int_0^\infty \int_0^\infty \frac{x^{\mu-1}}{(t^2+x^2)^{n+1}} \frac{1}{(t+1)} \; dt\, dx =$$

$$\frac{1}{2} \frac{(-1)^n \pi\, \Gamma\left(\frac{\mu}{2}\right)}{n!\, \sin\left(\frac{\mu}{2}\pi\right) \Gamma\left(\frac{\mu}{2}-n\right)} \; \frac{\pi}{\sin((\mu-2n-1)\,\pi)},$$

$$0<\mu<2n+2,\; n=0,\,1,\,2,\cdots .$$

Further, from Eq. (18) with $m = n = 0$:

$$\int_0^\infty \int_0^\infty \frac{x^{\mu-1}}{(t^2+x^2)} \frac{1}{(t^\nu+a)} \; dt\, dx = \frac{1}{2\nu} \frac{\pi}{\sin\left(\frac{\mu}{2}\pi\right)} \; \frac{\pi}{\sin\left(\left(\frac{\mu-1}{\nu}\right)\pi\right)} a^{(\mu-1)/\nu-1},$$

$$0<\mu<2,\; 0<\frac{\mu-1}{\nu}<1,\; a\neq 0.$$

3) $h(x) = x^{n+1}$, $G(t,\, x) = \frac{1}{(t^2+x^2)^r}$, $f(t) = \frac{t}{1+2t\cos\beta+t^2}$, $\alpha(t) = \ln t$.

In this case

$$I_1 = \int_0^\infty \frac{1}{(1+2t\cos\beta+t^2)}\left(\int_0^\infty \frac{x^n}{(t^2+x^2)^r}\,dx\right)dt$$

and

$$I_2 = \int_0^\infty x^n\left(\int_0^\infty \frac{1}{(t^2+x^2)^r}\frac{dt}{(1+2t\cos\beta+t^2)}\right)dx. \tag{19}$$

Applying Eq. (16) we obtain

$$I_1 = \frac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(r-\frac{n+1}{2}\right)}{\Gamma(r)}\int_0^\infty \frac{t^{n+1-2r}}{1+2t\cos\beta+t^2}\,dt, \quad 0 < n+1 < 2r$$

and by virtue from the result (Erdélyi, 1953)

$$\int_0^\infty \frac{x^{s-1}}{x^2+2ax\cos\theta+a^2}\,dx = -\pi a^{s-2}\csc\theta\csc(\pi s)\sin[(s-1)\theta],$$

$$a > 0, -\pi < \theta < \pi,\ 0 < \Re(s) < 2,$$

we have

$$I_1 = -\frac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(r-\frac{n+1}{2}\right)}{\Gamma(r)}\frac{\pi}{\sin[(n+2-2r)\pi]}\frac{\sin[(n+1-2r)\beta]}{\sin\beta}, \tag{20}$$

$$0 < n+1 < 2r, -\pi < \beta < \pi,\ 0 < n+2-2r < 2.$$

Then from Eqs. (19) and (20)

$$\int_0^\infty\int_0^\infty \frac{x^n}{(t^2+x^2)^r}\frac{1}{(1+2t\cos\beta+t^2)}\,dt\,dx =$$

$$I_1 = -\frac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(r-\frac{n+1}{2}\right)}{\Gamma(r)}\frac{\pi}{\sin[(n+2-2r)\pi]}\frac{\sin[(n+1-2r)\beta]}{\sin\beta},$$

$$0 < n+1 < 2r, -\pi < \beta < \pi,\ 0 < n+2-2r < 2.$$

4) $h(x) = \sin mx, \quad G(t,\, x) = \frac{1}{t^2+x^2}, \quad f(t) = \frac{\cos(b\sqrt{t})}{t\sqrt{t}}, \quad \alpha(t) = t^2.$

$$I_1 = 4\int_0^\infty \frac{\cos(b\sqrt{t})}{\sqrt{t}}\left(\int_0^\infty \frac{x\sin mx}{(t^2+x^2)}dx\right)dt$$

$$I_2 = 4\int_0^\infty x\sin mx\left(\int_0^\infty \frac{1}{(t^2+x^2)}\frac{\cos(b\sqrt{t})}{\sqrt{t}}dt\right)dx. \qquad (21)$$

By virtue of Eq. (13) we have

$$I_1 = 2\pi\int_0^\infty \frac{e^{-mt}\cos(b\sqrt{t})}{\sqrt{t}}dt,$$

which after of a simple change of variable and using (Erdélyi, 1953)

$$\int_0^\infty e^{-ax^2}\cos(xy)dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\;e^{-y^2/4a}, \quad \Re(a) > 0, y > 0 \qquad (22)$$

can be written as

$$I_1 = 4\pi\int_0^\infty e^{-mt^2}\cos(bt)dt = 2\pi\sqrt{\frac{\pi}{m}}\;e^{-b^2/4m}, \quad m > 0, b > 0. \qquad (23)$$

Finally, from Eqs. (21) and (23)

$$\int_0^\infty x\sin mx\left(\int_0^\infty \frac{1}{(t^2+x^2)}\frac{\cos(b\sqrt{t})}{\sqrt{t}}dt\right)dx = \frac{\pi}{2}\sqrt{\frac{\pi}{m}}\;e^{-b^2/4m}, \quad m > 0, b > 0,$$

that is,

$$\int_0^\infty x\sin mx\left(\int_0^\infty \frac{\cos(bt)}{(t^4+x^2)}dt\right)dx = \frac{\pi}{4}\sqrt{\frac{\pi}{m}}\,e^{-b^2/4m}, \quad m > 0, b > 0.$$

5) $h(x) = x^{m+1}, \quad G(t,\, x) = \frac{1}{(t^n+x^n)^r}, \quad f(t) = \frac{t}{(1+\beta\, t)^\nu}, \quad \alpha(t) = \ln t.$

$$I_1 = \int_0^\infty \frac{1}{(1+\beta t)^\nu}\left(\int_0^\infty \frac{x^m}{(t^n+x^n)^r}dx\right)dt$$

and

$$I_2 = \int_0^\infty x^m \left(\int_0^\infty \frac{1}{(t^n + x^n)^r} \frac{1}{(1 + \beta t)^\nu} dt \right) dx.$$

Using Eq. (16)

$$I_1 = \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) \Gamma\left(r - \frac{m+1}{n}\right)}{\Gamma(r)} \int_0^\infty \frac{t^{m+1-nr}}{(1 + \beta t)^\nu} dt, \quad 0 < \frac{m+1}{n} < r, n \neq 0.$$

The known result (Gradshteyn & Ryzhik, 1965; Prudnikov *et al.*, 1992)

$$\int_0^\infty \frac{x^{\mu-1}}{(1 + \beta x)^\nu} dx = \beta^{-\mu} B(\mu, \nu - \mu)$$

$$|\arg \beta| < \pi, \Re(\nu) > \Re(\mu) > 0,$$

leads us to

$$I_1 = \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) \Gamma\left(r - \frac{m+1}{n}\right)}{\Gamma(r)} \beta^{nr-m-2} B(m + 2 - nr,\ \nu + nr - m - 2)$$

$$0 < \frac{m+1}{n} < r,\ n \neq 0,\ |\arg \beta| < \pi, \Re(\nu) > m + 2 - nr > 0.$$

Finally,

$$\int_0^\infty x^m \left(\int_0^\infty \frac{1}{(t^n + x^n)^r} \frac{1}{(1 + \beta t)^\nu} dt \right) dx =$$

$$\frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) \Gamma\left(r - \frac{m+1}{n}\right)}{\Gamma(r)} \beta^{nr-m-2} B(m + 2 - nr,\ \nu + nr - m - 2)$$

$$0 < \frac{m+1}{n} < r,\ n \neq 0,\ |\arg \beta| < \pi, \Re(\nu) > m + 2 - nr > 0.$$

6) $h(x) = x^\nu e^{-ax^2}, \quad G(t,\ x) = \cos(xt), \quad f(t) = e^{t^2/4a} \cos t, \quad \alpha(t) = t^{1-\nu}.$

$$I_1 = (1 - \nu)^2 \int_0^\infty t^{-\nu} e^{t^2/4a} \cos t \left(\int_0^\infty e^{-ax^2} \cos(xt) dx \right) dt$$

$$I_2 = (1-\nu)^2 \int_0^\infty e^{-ax^2} \left(\int_0^\infty t^{-\nu} \cos(xt)\; e^{t^2/4a} \cos t\; dt \right) dx.$$

In view of Eq. (22) we can write

$$I_1 = \frac{(1-\nu)^2}{2} \sqrt{\frac{\pi}{a}} \int_0^\infty \frac{\cos t}{t^\nu} dt,$$

now evaluating this integral by means of (Erdélyi, 1953)

$$\int_0^\infty \frac{\cos(xy)}{x^\nu} dx = \frac{\pi y^{\nu-1}}{2\Gamma(\nu)\cos(\nu\pi/2)}, \quad 0 < \nu < 1, y > 0$$

we get

$$I_1 = \frac{(1-\nu)^2}{4} \sqrt{\frac{\pi}{a}} \frac{\pi}{\Gamma(\nu)\cos(\nu\pi/2)}, \quad 0 < \nu < 1.$$

Therefore,

$$\int_0^\infty e^{-ax^2} \left(\int_0^\infty t^{-\nu} \cos(xt) e^{t^2/4a} \cos t\; dt \right) dx = \frac{1}{4} \sqrt{\frac{\pi}{a}} \frac{\pi}{\Gamma(\nu)\cos(\nu\pi/2)}, \; 0 < \nu < 1.$$

ACKNOWLEDGEMENT

H. Prado is partially supported by Project DICYT USACH # 041133PC.

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Submitted : *31/1/2012*
Revised : *27/6/2012*
Accepted : *22/7/2012*

بعض النتائج حول محول لابلاس - ستيتجز

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خلاصة

نقدم في هذا البحث مبرهنتين حول محول لابلاس - ستيتجز المبرهنة الاولى هي مبرهنة تلاف معممة. أما الثانية فهي تعميم لنظرية برسيفال وذلك لمؤثر k(s,t). ويمكن استخراج العديد من الحالات الخاصة من هاتين المبرهنتين. كما نخصص جزء من هذا البحث لتطبيقات هاتين المبرهنتين وذلك لحساب بعض التكاملات.