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On determinantal recurrence relations of banded matrices

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Abstract

We provide an algorithm based on a less-known result about recurrence relations for the determinants of banded matrices. As a consequence, we prove recent conjectures on the determinants of particular classes of pentadiagonal matrices and simple alternative proofs for other results.

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1. Introduction

The standard tridiagonal matrices are square matrices of the form

$$\begin{pmatrix} * & * & & & \\ * & * & * & & \\ & * & \ddots & \ddots & \\ & & \ddots & \ddots & * \\ & & & * & * \end{pmatrix}, \tag{1}$$

where the non-mentioned entries should be read as zero. Analogously, we have the pentadiagonal matrices which are of the form

$$\begin{pmatrix} * & * & * & & & & \\ * & * & * & * & & & \\ * & * & \ddots & \ddots & \ddots & & \\ & * & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & * \\ & & & \ddots & \ddots & \ddots & * \\ & & & & * & * & * \end{pmatrix}. \tag{2}$$

These matrices (particularly, those which are Toeplitz) occur in many numerical problems involving different kinds of differential equations and they also naturally emerge in many areas of pure, applied, and numerical mathematics, engineering, statistics, signal processing, among many others, with particular emphasis to the computational problems related to the calculation of spectra, determinant, permanent, characteristic polynomial, inverse, power, and different types of decompositions (Arbenz, 1991; Arıkan & Kılıc, 2017; Cinkir, 2012; Diele & Lopez, 1998; Elouafi, 2013; Elouafi, 2011; Hadj & Elouafi, 2008; Jia *et al.*, 2016; Kratz, 2010; Kratz & Tentler, 2008; Kılıc & El-Mikkawy, 2008; Montaner &

Alfaro, 1995; Marr & Vineyard, 1988; Sweet, 1969). There are many extensions of these matrices which deserved attention in many areas of research as we can find, for example, in (Anđelić & da Fonseca, 2021; Arbenz, 1991; Egerváry & Szász, 1928; da Fonseca & Yılmaz, 2015; Kratz, 2010; Kratz & Tentler, 2008; Losonczi, 1992; McMillen, 2009; Ohashi *et al.*, 2015; Takahira *et al.*, 2019). All of them belong to the family of the *banded matrices*.

Definition 1.1. *Let r and s be nonnegative integers. A square matrix $A = (a_{i,j})_{i,j\geqslant 0}$ is called an (r, s)-* banded matrix *if $a_{i,j} = 0$ unless $-s \leqslant i - j \leqslant r$.*

The bandwidth of an (r, s)-banded matrix is $r + s + 1$. For example, any tridiagonal matrix of the form (1) is a $(1, 1)$-banded matrix, while the pentadiagonal matrix (2) is $(2, 2)$-banded, with bandwidths, respectively 3 and 5. Another example is an $n \times n$ Hessenberg matrix. Here the bandwidth is $n + 1$.

Since we are mainly interested on the discussion of the determinants of certain banded matrices, it is useful to recall the next theorem.

Theorem 1.1. *(Zakrajšek & Petkovšek, 2004, Theorem 2) Let $A = (a_{i,j})_{i,j\geqslant 0}$ be an (r, s)-banded matrix and $\delta = \binom{r+s}{r}$. Denote by d_n the principal minor of A consisting of the entries indexed by the rows and columns $0, 1, \ldots, n$. Then, for $n \geqslant \delta$, the sequence $(d_n)_{n\geqslant 0}$ satisfies a nontrivial homogeneous linear recurrence of the form*

$$d_n = \sum_{n=1}^{\delta} R_k \, d_{n-k} \,,$$

where each R_k is a homogeneous rational function of degree k of entries $a_{n-i,n-j}$, with $0 \leqslant i \leqslant \delta - 1$ and $-s \leqslant j \leqslant r + \delta - 1$.

Recall that a rational function f is *homogeneous of degree k*, if $f(tx_1, \ldots, tx_n) = t^k f(x_1, \ldots, x_n)$.

This theorem and its proof are largely unknown, for example, in the linear algebra community, and both can be quite useful in many instances.

In this note, we describe a Mathematica (Wolfram Research, Inc., 2020) algorithm based on Theorem 1.1 which allows us to provide alternative proofs to some recent results and answers to two conjectures.

2. The algorithm

In this section we provide the implementation of the main Mathematica functions, based on Theorem 1.1, that allowed us to compute the results contained in the following sections.

In the listing below, the function `equations` constructs the $\delta = \binom{r+s}{r}$ equations associated with the determinants of a general (r, s)-banded matrix with elements $\mathtt{a[i,j]}$. These equations are used by function `findRecurrenceEquation` to compute the corresponding recurrence relation

$$\mathtt{d[1,n]} == \alpha_1 \mathtt{d[1,n-1]} + \cdots + \alpha_\delta \mathtt{d[1,n-\delta]}\,.$$

Listing 1. Main functions

```
computeSubsets[r_, s_] :=
  Block[{i = 0},
    Clear[indexOf, subset];
    Do[indexOf@x = ++i;
    subset@i = x , {x,
      Subsets[Range[r + s], {r}]}]];

Clear[a, d, n]; Protect[a, d, n];

dd[n_, k_, r_, s_] :=
  Block[{list, jr},
  list = subset[k];
  jr = Last@list;
  If[jr == r + s,
    a[n, n + s]*d[indexOf[Flatten[
```

```
    {1, Most[list] + 1}]], n - 1]
   ,
  (-1)^r * a[n, n - r] *
   d[indexOf[list + 1], n - 1] +
   Sum[(-1)^(r + i) * a[n, n - r + list[[i]]]*
    d[indexOf[Flatten[{1,
       Drop[list, {i}] + 1}]], n - 1], {i, r}]
   ]];

equations[n_, r_, s_] :=
  Block[{delta = Binomial[r + s, r]},
   computeSubsets[r, s];
   Flatten@Table[d[i, n - j + 1] ==
     dd[n - j + 1, i, r, s], {i, delta},
       {j, delta}]];

variables[eqs_] := DeleteDuplicates[Cases[eqs,
  d[_, _], Infinity]];

variablesToEliminate[eqs_] :=
   DeleteDuplicates[Cases[eqs,
     d[i_, _] /; i != 1, Infinity]];

findRecurrenceEquation[eqs_] :=
  Block[{sol},
   sol = Eliminate[eqs,
     variablesToEliminate@eqs];
   Reduce[ If[Head@sol === And,
     SelectFirst[sol, MemberQ[#, d[1, n],
       Infinity] &], sol], d[1, n]]];
```

Some remarks are in order. First the number of equations necessary to compute the recurrence relation for the determinants of an (r, s)-banded matrix increases very rapidly with the bandwidth $r+s+1$ of the matrix. Thus, the process of variable elimination used by function `findRecurrenceEquation` can be quite time-consuming. Also, for a general banded matrix, the coefficients $\alpha_{\mathtt{i}}$ in the recurrence relation can be large and complex expressions.

Given a particular banded matrix with concrete elements $b[i, j]$, it is better to start by transforming the general equations into the particular ones before computing the recurrence relation:

```
eqs = equations[n,r,s]/.{a[i_,j_]:>b[i,j]};
findRecurrenceEquation[eqs];
```

Mathematica's `Eliminate` function, used in `findRecurrenceEquation`, can be very slow when applied to equations computed for general matrices containing several parameters. One way to overcome this problem is by first computing the matrix associated with the linear system with a clever ordering of the variables and then perform row reduction of this matrix. An implementation of this method is next listed.

Listing 2. An improved recurrence finder

```
findRecurrenceEquation2[eqs_] :=
  Block[{vars, m, lastRow},
   vars = Sort[variables[eqs],
     (First[#1] > First[#2]) ||
     ((First[#1] > First[#2]) &&
      Assuming[Element[n, Integers],
        Simplify[Last[#1] < Last[#2]]]) &];
   m = CoefficientArrays[eqs, vars][[2]];
   lastRow = Last[RowReduce[m]];
   Reduce[Dot[lastRow, vars] == 0, d[1, n]]];
```

Finally, an explicit formula for the determinants of a banded matrix can be obtained from the computed recurrence relation by using Mathematica's function `RSolve`.

In the next sections we will find some applications of this algorithm.

3. Recurrence relations

For more than five decades there has been an effort for investigating the structure of the determinant of pentadiagonal matrices in order to develop efficient methods for determining they spectra. We believe that R.A. Sweet first determined in 1969 a recursive formula for the determinant of any matrix of the form (2) and, consequently, a method to determine explicitly and locate the eigenvalues of such matrices (Sweet, 1969). For Toeplitz matrices

$$P_n = \begin{pmatrix} a & b & c & & & & \\ d & a & b & c & & & \\ e & d & \ddots & \ddots & \ddots & & \\ & e & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & c \\ & & & \ddots & \ddots & \ddots & b \\ & & & & e & d & a \end{pmatrix}_{n\times n},$$

Sweet's 7-term recurrence relation (Sweet, 1969, (10)) reads as

$$p_k = ap_{k-1} + (ec - bd)p_{k-2} + (cd^2 + b^2e - 2ace)p_{k-3} + ce(ce - bd)p_{k-4} + a(ce)^2 p_{k-5} - (ce)^3 p_{k-6}\,,$$

for $k \geqslant 5$, with initial conditions $p_{-1} = 0$, $p_0 = 1$ and $p_k = \det P_k$, for $k = 1, 2, 3, 4$. This recurrence relation has been re-discovered in many instances, as in (Arbenz, 1991; Cinkir, 2012; Jia *et al.*, 2016; Kratz & Tentler, 2008; Montaner & Alfaro, 1995), and was not always properly acknowledged.

We will provide the recent examples where we can find as an immediate application of the algorithm proposed in Section 2.

Recently, (Küçük & Düz, 2017, Remark 9) claimed that the determinant of the 3-tridiagonal matrix

$$T_n = \begin{pmatrix} a & 0 & 0 & b & & & \\ 0 & a & 0 & 0 & b & & \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \\ c & 0 & \ddots & \ddots & \ddots & \ddots & b \\ & c & \ddots & \ddots & \ddots & \ddots & 0 \\ & & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & c & 0 & 0 & a \end{pmatrix} \tag{3}$$

cannot be written in terms of a linear recurrence relation involving the determinants of its minors for $a = 2x$, $b = 1$ and $c = -1$ (see also (da Fonseca, 2018b)). This is clearly wrong since, from Theorem 1.1, with $r = s = 3$, there is a recurrence relation $\delta + 1 = \binom{6}{3} + 1 = 20 + 1$ terms for the d_n. Applying the algorithm, this recurrence is

$$d_n = ad_{n-1} - bcd_{n-2} + abcd_{n-3} - a^2bcd_{n-4} + ab^2c^2d_{n-5} - b^3c^3d_{n-6} + ab^3c^3d_{n-7} - b^4c^4d_{n-8}.$$

In matter of fact, due to the pattern of T_n in this particular case, we may recover the exact formula for the recurrence relation established in (Trojovský & Zvoníková, 2019), where the number of terms can be significantly reduced.

Theorem 3.1. *Let $n > 8$ be an integer and $d_n = \det T_n$, with $a = 2x$, $b = 1$ and $c = -1$. Then the sequence $(d_n)_{n>8}$ satisfies the recurrence relation with 9 terms of the form*

$$d_n = 2xd_{n-1} - d_{n-2} + 2xd_{n-3} - 4x^2d_{n-4} + 2xd_{n-5} - d_{n-6} + 2xd_{n-7} - d_{n-8}\,,$$

with initial conditions

$$\begin{aligned} d_1 &= 2x,\ d_2 = 4x^2,\ d_3 = 8x^3\,, \\ d_4 &= 4x^2(4x^2-1),\ d_5 = 2x(4x^2-1)^2\,, \\ d_6 &= (4x^2-1)^3,\ d_7 = 4x(2x^2-1)(4x^2-1)^2\,, \\ d_8 &= (4x)^2(2x^2-1)^2(4x^2-1)\,. \end{aligned}$$

From (da Fonseca & Yılmaz, 2015; Losonczi, 1992), one can deduce a compact formula for Theorem 3.1:

$$d_n(x) = (\sqrt{bc})^n U_{q+1}^r\left(\frac{x-a}{2\sqrt{bc}}\right) U_q^{3-r}\left(\frac{x-a}{2\sqrt{bc}}\right)\,,$$

where

$$n = 3q + r\,, \quad \text{with } r \in \{0,1,2\}\,,$$

and $\{U_n(x)\}_{n\geqslant 0}$ is the sequence of Chebyshev polynomials of the second kind.

Another approach can be found is in (da Silva, 2017) for the general case (3), where we can see that the formulas become quite intricate.

4. Two conjectures

Another recent and interesting problems were proposed in (Anđelić & da Fonseca, 2021) involving the determinants of an extension to pentadiagonal matrices. It was conjectured the following.

Conjecture 1. *The determinant of the matrix*

$$A_n = \begin{pmatrix} 1 & 1 & & & & & 1 \\ 1 & 1 & 1 & & & & \\ 1 & 1 & \ddots & \ddots & & & \\ & 1 & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & 1 \\ & & & & 1 & 1 & 1 \end{pmatrix}_{n\times n}$$

is, for $n \geqslant 4$,

$$\det A_n = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4 \\ 2 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 2 \pmod 4 \\ 0 & \text{if } n \equiv 3 \pmod 4 \end{cases}\,.$$

Conjecture 2. *The determinant of the matrix*

$$B_n = \begin{pmatrix} 1 & 1 & & & & 1 & \\ 1 & 1 & 1 & & & & 1 \\ 1 & 1 & \ddots & \ddots & & & \\ & 1 & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & 1 \\ & & & & 1 & 1 & 1 \end{pmatrix}_{n\times n}$$

is, for $n \geqslant 4$,

$$\det B_n = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ 2 & \text{if } n \equiv 1 \pmod 4 \\ 3 & \text{if } n \equiv 2 \pmod 4 \\ 1 & \text{if } n \equiv 3 \pmod 4 \end{cases} .$$

Both conjectures have recently attracted the attention of many researchers (Amanbek *et al.*, 2020; Kurmanbek *et al.*, 2020; Shitov, 2021).

Using the algorithm developed in Section 2, we are able here to confirm these conjectures by providing a compact formula for both. First we prove an auxiliary result.

Lemma 4.1. *Let F_n be the all 1's $(2,1)$-banded matrix of order n, that is, defined by $f_{i,j} = 1$ if $-1 \leq i-j \leq 2$, and 0 elsewhere. Then the sequence of determinants of F_n satisfies the recurrence relation*

$$d_n = d_{n-4} \tag{4}$$

with initial conditions $d_1 = d_4 = 1$ and $d_2 = d_3 = 0$. Moreover, for $n \geq 1$, this recurrence has the explicit solution

$$d_n = \frac{1}{2}\left(\sin\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{2}\right) + 1\right) .$$

Proof. Equation (4) was obtained by using the algorithm presented in Section 2. The explicit solution was obtained by using Mathematica's `Rsolve` function. □

Next we are able to prove both conjectures.

Theorem 4.2. *The sequence of determinants of A_n, $(d_n)_{n\geq 3}$, satisfies explicitly*

$$d_n = \frac{1}{2}\left(1-(-1)^n\right) + \sin\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{2}\right) .$$

Proof. Developing the determinant of A_n from its first row, we get

$$\det A_n = \det A_n^{(1,1)} - \det A_n^{(1,2)} + (-1)^{n+1} \det A_n^{(1,n)}.$$

$A_n^{(1,1)}$ is a full, all 1's, order $n-1$, $(2,1)$-banded matrix. From Lemma 4.1, the general recurrence relation for the determinant of an order n matrix of this kind is (4). Thus the associated characteristic polynomial is $p_1(x) = x^4 - 1$.

The matrix $A_n^{(1,2)}$ is a $(2,1)$-banded matrix of order $n-1$ where $a_{3,1} = 0$ and $a_{i,j} = 1$ when $-1 \leq i-j \leq 2$ and $(i,j) \neq (3,1)$. By using the algorithm presented in Section 2, we conclude that the general recurrence relation for the determinant of an order n matrix of this kind is also (4). Therefore, the associated characteristic polynomial is $p_2(x) = p_1(x)$.

Similarly, $A_n^{(1,n)}$ is a $(1,2)$-banded matrix of order $n-1$ where $a_{i,j} = 1$ when $-2 \leq i-j \leq 1$. Noticing that $A_n^{(1,n)}$ is the transpose of $A_n^{(1,1)}$, we get that the general recurrence relation for the determinant of an order n matrix of this kind is also (4).

Now, notice that the recurrence relation for the sequence $d'_n = (-1)^n d_n$, where d_n is given by (4), is also $d'_n = d'_{n-4}$ with associated characteristic polynomial given by $p_3(x) = p_1(x)$.

We can now conclude that the characteristic polynomial associated to the recurrence $d_n = \det A_n$ is given by $\operatorname{lcm}(p_1(x), p_2(x), p_3(x)) = x^4 - 1$, hence $d_n = d_{n-4}$ for $n > 4$.

Finally, the explicit solution was obtained by using Mathematica's function. □

Corollary 4.3. *The sequence of determinants of A_n, $(d_n)_{n\geq 1}$, satisfies the recurrence relation* (4), *with initial conditions $d_1 = 2$, $d_2 = -1$, $d_3 = 0$, and $d_4 = 1$.*

Theorem 4.4. *The sequence of determinants of* B_n, $(d_n)_{n\geq 4}$, *satisfies the recurrence relation* (4), *with initial conditions* $d_1 = 2$, $d_2 = 3$, $d_3 = 1$, *and* $d_4 = 0$. *Moreover, for* $n \geq 1$, *this recurrence has the explicit solution*

$$\det B_n = \frac{1}{2}\left(\sin\left(\frac{n\pi}{2}\right) - 3\cos\left(\frac{n\pi}{2}\right) + 3\right) . \tag{5}$$

Proof. Let us start expanding the determinant of B_n along the first row, i.e.,

$$\det B_n = \det B_n^{(1,1)} - \det B_n^{(1,2)} + (-1)^n \det B_n^{(1,n-1)} . \tag{6}$$

Notice that $B_n^{(1,1)}$ is equal to matrix A_{n-1} in the previous theorem. Thus the recurrence relation for the determinants of $B_n^{(1,1)}$ is (4).

Next, take $C_{n-1} = B_n^{(1,2)}$ and expand the determinant of C_{n-1} along the first row:

$$\det C_{n-1} = \det C_{n-1}^{(1,1)} - \det C_{n-1}^{(1,2)} + (-1)^n \det C_{n-1}^{(1,n-1)} .$$

The matrix $C_{n-1}^{(1,1)}$ is of the same kind as matrix $A_n^{(1,1)}$ in the proof of Theorem 4.2 (a $(2,1)$-banded matrix of order $n-2$). Hence it also satisfies the recurrence (4). On the other hand, the matrix $C_{n-1}^{(1,2)}$ is changed into $(2,1)$-banded of order $n-3$ after expanding along the first column. The general recurrence relation for the determinant of this kind of matrix is again (4). The matrix $C_{n-1}^{(1,n-1)}$ becomes $(1,2)$-banded of order $n-3$ after expanding along the first column. The determinants of these matrices also satisfy the general recurrence relation (4).

It is now simple to conclude that the determinants of $B_n^{(1,2)}$ also satisfy (4).

Next, we consider $E_{n-1} = B_n^{(1,n-1)}$ and expand the determinant of E_{n-1} along the last column. We get

$$\det E_{n-1} = (-1)^n \det E_{n-1}^{(1,n-1)} - \det E_{n-1}^{(n-2,n-1)} + \det E_{n-1}^{(n-1,n-1)} .$$

Now, $E_{n-1}^{(1,n-1)}$ is an upper triangular matrix with main diagonal all equal to 1. Expanding the determinant of $E_{n-1}^{(n-2,n-1)}$ along the last row, we obtain a full $(1,2)$-banded matrix of order $n-3$ with banded elements all equal to 1. Therefore, from Lemma 4.1, the sequence of determinants satisfies (4). The matrix $E_{n-1}^{(n-1,n-1)}$ is also a full $(1,2)$-banded matrix of order $n-2$ with banded elements all equal to 1. Thus the sequence of determinants satisfies (4). It is now simple to conclude that the sequence of determinants for E_{n-1} also satisfies (4).

Recalling (6) we conclude that the sequence of determinants of B_n satisfies $d_n = d_{n-4}$. □

Corollary 4.5. *The sequence of determinants of* B_n, $(d_n)_{n\geq 1}$, *satisfies the recurrence relation* (4), *with initial conditions* $d_1 = 2$, $d_2 = 3$, $d_3 = 1$, *and* $d_4 = 0$.

From both corollaries, we guess that the recurrence relation (4) is indeed satisfied for a larger (even much larger) family of pentadiagonal matrices. We leave this as an open research problem.

For other related recurrences on other families of matrices, the reader is referred, for example, to (da Fonseca, 2018a).

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