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Automorphisms on complex simple Lie algebras of order 3

Ching-I Hsin*
Dept. of Multimedia and Game Development, Minghsin University of Science and Technology
Hsinchu County, Taiwan.
**Corresponding author: hsin@must.edu.tw*

Abstract

For complex simple Lie algebras, the article provides the classification of all automorphisms of order 3. The method is an extension of Dynkin diagrams so that the classification is a listing of diagrams that represent automorphisms of order 3. This work extends an earlier result on automorphisms of order 2. As an application, it shows that for automorphisms of orders 2 and 3 only, the invariant subalgebra determines the automorphism.

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1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra (Humphreys, 1973). Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\Delta \subset \mathfrak{h}^*$ be its root system. It leads to a root space decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\Delta} \mathfrak{g}_\alpha .$$

Let $\Pi \subset \Delta$ be a simple system, with the lowest root φ. Let D^1 be the extended Dynkin diagram of $\mathfrak{g}$ so that its vertices are members of $\Pi \cup \{\varphi\}$. The roots $\Pi \cup \{\varphi\}$ are linearly dependent, and there are unique relatively prime positive integers $\{a_\alpha\}_{D^1}$ such that $\sum_{D^1} a_\alpha \alpha = 0$ (Kac, 1990). Here relatively excellent means that they have no common factor other than 1.

A Kac diagram c is an assignment of relatively prime nonnegative integers $\{c_\alpha\}_{D^1}$. Given a Kac diagram c, we let $n = \sum_{D^1} a_\alpha c_\alpha$ and let $\omega_n = \exp(2\pi i/n) \in \mathbb{C}$. We say that c represents a $\mathfrak{g}$-automorphism θ: $\mathfrak{g} \rightarrow \mathfrak{g}$ if

$$\theta(X)=\begin{cases} X & \text{for } X\in\mathfrak{h},\\ \omega^{c_\alpha}X & \text{for } X\in\mathfrak{g}_\alpha \text{ and } \alpha\in \mathrm{D}^1. \end{cases}$$

In this case, θ has order n, because n it is the smallest positive integer such that $\theta^n = 1$.

Two $\mathfrak{g}$-automorphisms θ, θ' are said to be equivalent if there is a $\mathfrak{g}$-automorphism σ such that $\theta' = \sigma\theta\sigma^{-1}$. The list of all $\mathfrak{g}$-automorphisms of order two up to equivalence has been given (Chuah, 2012, Figure 1-3). Automorphisms of order 2 play essential roles in the study of real semisimple Lie algebras and symmetric spaces. This method is extended here to apply automorphisms of order 3 for generalized symmetric spaces, thereby establishing the importance of automorphisms of order 3 (Wolf & Gray, 1968). In this article, we provide a diagrammatic classification of all $\mathfrak{g}$-automorphisms of order 3.

The group of all $\mathfrak{g}$-automorphisms, denoted by aut($\mathfrak{g}$), has a natural topology structure. We say that $\theta \in$ aut($\mathfrak{g}$) is inner if it lies in the connected component, which contains the identity map. Otherwise, we say that θ is outer.

Theorem 1.1. *Up to equivalence, all inner $\mathfrak{g}$-automorphisms of order three are represented by the Kac diagrams in Figure 1 for classical Lie algebras and Figure 2 for exceptional Lie algebras.*

In these Kac diagrams, we make the convention that the vertices α such that $c_\alpha = 0$ are drawn without indicating the integer 0. For example, in the first diagram of Figure 2, two vertices satisfy $c_\alpha = 1$, and five vertices satisfy $c_\alpha = 0$.

The complex simple Lie algebras are classified by their Dynkin diagrams, which are A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 (see Section 11.4 of Humphreys, 1973). In particular, the Lie algebras of A_n, B_n, C_n, D_n are constructed (see Section 1.2 of Humphreys, 1973).

There are more complicated diagrams, known as twisted diagrams. For the Lie algebra D_4, there is a twisted diagram D^3, which we will explain later in (3).

Theorem 1.2. *Up to equivalence, there are only two outer $\mathfrak{g}$-automorphisms of order 3. They occur when $\mathfrak{g} = D_4$ and are represented by the twisted diagrams in Figure 3.*

The fixed points of mappings are studied in many areas of mathematics, see for example (Pant, Chauhan, Cho & Gordji, 2015). We also study the fixed points here. Given a $\mathfrak{g}$-automorphism θ, we let $\mathfrak{g}^\theta$ denote its fixed point set, also known as invariant subalgebra, namely

$$\mathfrak{g}^\theta = \{X \in \mathfrak{g};\ \theta(X) = X\}. \tag{1}$$

As an application of the lists of diagrams, we study to what extent $\mathfrak{g}^\theta$ determine θ. It is known that for automorphisms of order 2, the isomorphic class of $\mathfrak{g}^\theta$ determines the equivalence class of θ. The following theorem says that this is also true for automorphisms of order 3, but not higher order automorphisms. The theorem $\cong$ denotes Lie algebra isomorphism.

Theorem 1.3. *Let θ, σ be $\mathfrak{g}$-automorphisms of order n.*
(a) For $n = 2, 3$, θ and σ are equivalent if and only if $\mathfrak{g}^\theta \cong \mathfrak{g}^\sigma$.
(b) For $n \geq 4$, the condition, $\mathfrak{g}^\theta \cong \mathfrak{g}^\sigma$ does not imply that θ and σ are equivalent.

We shall prove Theorems 1.1 and 1.2 in Section 2 and prove Theorem 1.3 in Section 3.

2. Automorphisms of order 3

Recall that $\mathfrak{g}$ is a finite-dimensional complex simple Lie algebra, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $\Delta \subset \mathfrak{h}^*$ is its root system, and

$$\mathfrak{g} = \mathfrak{h} + \sum_{\Delta} \mathfrak{g}_\alpha$$

is the root space decomposition.

We recall a result of Kac on finite order $\mathfrak{g}$-automorphisms. Let $\Pi \subset \Delta$ be a simple system. Let $\sigma \in \mathrm{aut}(\Delta)$ be a root automorphism of order k which stabilizes Π, namely $\sigma(\Pi) = \Pi$. Clearly, σ can be the identity. Otherwise, the list of Dynkin diagrams show that $\sigma \neq 1$ occurs only in the following cases,

$k = 1$: any $\mathfrak{g}$,
$k = 2$: $\mathfrak{g} = A_n$ ($n \geq 2$), D_n, E_6, (2)
$k = 3$: $\mathfrak{g} = D_4$.

Let $\mathbb{Z}_k$ denote the group of integers modulo k. There exists a Lie algebra automorphism of order k, still denoted by σ, such that

$$\sigma \mathfrak{g}_\alpha = \mathfrak{g}_{\sigma\alpha}$$

for all $\alpha \in \Delta$. It leads to a $\mathbb{Z}_k$-grading $\mathfrak{g} = \sum_{\mathbb{Z}_k} \mathfrak{g}_r$, where $\mathfrak{g}_r$ has the eigenvalue $\exp(2r\pi i/k)$ with respect to σ. Then $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$, and it acts on $\mathfrak{g}_r$ for all $r \in \mathbb{Z}_k$. We construct a diagram.

D^k = set of vertices representing $\Pi_0 \cup \{\varphi\}$, (3)

where Π_0 is a simple system of $\mathfrak{g}_0$, and φ is the lowest weight of the $\mathfrak{g}_0$-representation on $\mathfrak{g}_1$. Also, the edges of D^k are drawn according to the usual rule of Dynkin diagrams. Note that if σ is the identity, so that $k = 1$, then $\mathfrak{g} = \mathfrak{g}_0 = \mathfrak{g}_1$, so φ is just the lowest root of $\Pi = \Pi_0$. In this case, D^1 is just the extension of the Dynkin diagram of $\mathfrak{g}$.

Having added φ to Π_0, it follows that the roots represented by the vertices of D^k are linearly dependent, so there are unique relatively prime positive integers $\{a_\alpha\}_{D^k}$ such that

$$\sum_{D^k} a_\alpha \alpha = 0 .$$

These integers are presented in Chapter 4, Tables Aff 1, Aff 2, and Aff 3 (Kac, 1990).

A Kac diagram c an assignment of relatively prime nonnegative integers to D^k, namely c_α is a nonnegative integer for each $\alpha \in D^k$, and $\{c_\alpha\}_{D^k}$ is a set of relatively prime integers. Two Kac diagrams on D^k are said to be equivalent if they are related by a diagram symmetry of D^k. If n is a positive integer, we let $\omega_n = \exp(2\pi i/n) \in \mathbb{C}$ be the primitive n-th root of unity.

Theorem 2.1. (Kac) (Helgason, 2001, Chapter X; Kac, 1990, Chapter 8) *There is a bijective correspondence between the equivalence classes of Kac diagrams and finite order* $\mathfrak{g}$-*automorphisms. In this correspondence, if c is a Kac diagram, it represents a* $\mathfrak{g}$-*automorphism* θ *of order* $n = k\sum_{D^k} a_\alpha c_\alpha$, *where there exist root vectors* $\{X_\alpha\}_{D^k}$ *such that* $\theta(X_\alpha) = \omega_n^{c_\alpha} X_\alpha$. *Here* $\mathfrak{g}^\theta$ *is reductive, its semisimple part has a Dynkin diagram* $\{\alpha \in D^k; c_\alpha = 0\}$, *and its center has dimension.*
$\#\{\alpha \in D^k; c_\alpha \neq 0\} - 1$.

The case where $n = 2$ has been classified in (Chuah, 2012, Figures 1-3). In this article, we classify the Kac diagrams $n = 3$ corresponding to all $\mathfrak{g}$-automorphisms of order 3. By Theorem 2.1, we look for all Kac diagrams c such that

$$k\sum_{D^k} a_\alpha c_\alpha = 3 . \qquad (4)$$

Recall that k can be 1,2,3 as discussed in (2). The condition (4) excludes $k = 2$, so we deal with two cases,

(a) $k = 1$ for any ,
(b) $k = 3$ for $\mathfrak{g} = D_4$. (5)

We start with (5)(a). Here (4) becomes $\sum_{D^1} a_\alpha c_\alpha = 3$, which leads to the following four cases:

(a) $a_\alpha = c_\alpha = a_\beta = c_\beta = a_\gamma = c_\gamma = 1$,
$c = 0$ on $D^1 \backslash \{\alpha, \beta, \gamma\}$,

(b) $a_\alpha = a_\beta = 1,\ c_\alpha = 1,\ c_\beta = 2,\ c = 0$ on $D^1\backslash\{\alpha, \beta\}$,
(c) $a_\alpha = 1,\ a_\beta = 2,\ c_\alpha = c_\beta = 1,$ (6)
$c = 0$ on $D^1\backslash\{\alpha, \beta\}$,
(d) $a_\alpha = 3,\ c_\alpha = 1,\ c = 0$ on $D^1\backslash\{\alpha\}$.

We make the convention that for the Kac diagrams c in Figures 1-3, the vertices α without indicated integer refer to $c_\alpha = 0$. Figures 1-2 provide all Kac diagrams which satisfy one of the conditions of (6). For example:
the first Kac diagram of A_n satisfies (6)(a),
the second Kac diagram of A_n satisfies (6)(b),
the first Kac diagram of B_n satisfies (6)(c),
the second Kac diagram of G_2 satisfies (6)(d).

We conclude that Figures 1-2 classifies all Kac diagrams with $k = 1$. Figure 1 deals with the classical Lie algebras A_n, B_n, C_n, D_n, and Figure 2 deals with the exceptional Lie algebras E_6, E_7, E_8, F_4, G_2.

Next, we consider (5)(b). In this case, $\mathfrak{g} = D_4$ and (4) becomes $\sum_{D^3} a_\alpha c_\alpha = 1$. So there is exactly one vertex $a_\alpha = c_\alpha = 1$,, and c assigns 0 to the remaining vertices of D^3. Such Kac diagrams are classified by Figure 3.

We conclude that, by Theorem 2.1, Figures 1-3 classify all the Kac diagrams for $\mathfrak{g}$-automorphisms of order 3.

Recall that a $\mathfrak{g}$-automorphism is called inner if it lies in the connected component of aut($\mathfrak{g}$) which contains the identity mapping and is called outer otherwise. In this diagrammatic classification, the automorphism is inner if and only if the diagram is D^1 (Chuah, 2012, (4.4)). Therefore, the automorphisms represented by Figures 1-2 are inner, and the automorphisms represented by Figure 3 are outer. This completes the proof for Theorems 1.1 and 1.2.

3. Invariant subalgebras

Given a Kac diagram c on D^k, we define its kernel as the subdiagram of D^k provided by

$$\ker c = \{\alpha \in D^k;\ c_\alpha = 0\}.$$

In the following proposition, isomorphism of diagrams is understood as bijection of vertices which preserves edge relations. Let $\ncong$ denote diagrams that are not isomorphic.

Proposition 3.1. *Let c, c' be Kac diagrams on* D^k*, which represent* $\mathfrak{g}$*-automorphisms of order 3.*

If c, c' are not equivalent, then ker c $\ncong$ *ker c'.*

Proof. This is proved by checking through the list of Kac diagrams in Figures 1-3. For example, consider the five Kac diagrams of $\mathfrak{g} = E_6$ in Figure 2. Their kernels are respectively the Dynkin diagrams of $A_5, A_1 + A_4, D_5, A_2 + A_2 + A_2, D_4$, so these kernels are not isomorphic to one another. Similarly, for $\mathfrak{g} = E_7$, Figure 2 provides five Kac diagrams whose kernels are respectively the Dynkin diagrams of $D_6, A_1 + D_5, A_6, A_2 + A_5, E_6$, so these kernels are not isomorphic to one another.

By checking through all the Kac diagrams in Figures 1-3, Proposition 3.1 follows. ■

We shall see that Proposition 3.1 fails for automorphisms of order ≥ 4, as Figure 4 provides a counterexample.

Recall that, for a $\mathfrak{g}$-automorphism θ, we let $\mathfrak{g}^\theta$ denote its invariant subalgebra (1). We now prove Theorem 1.3(a). Let θ, σ be $\mathfrak{g}$-automorphisms. If they are equivalent, then there exists a $\mathfrak{g}$-automorphism μ such that $\sigma = \mu\theta\mu^{-1}$. It implies that $\mu(\mathfrak{g}^\theta) = \mathfrak{g}^\sigma$, hence $\mathfrak{g}^\theta \cong \mathfrak{g}^\sigma$. So the remaining issue is to prove the "if" part of the statement. The $\mathfrak{g}$-automorphisms of order 2 are also known as involutions. The result for involutions is known in the literature nevertheless, we elaborate the argument here. There is a bijective correspondence between the equivalence classes of $\mathfrak{g}$-involutions θ and isomorphic classes of real forms $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$, given by θ stabilizes $\mathfrak{g}_\mathbb{R}$ and is a Cartan involution of $\mathfrak{g}_\mathbb{R}$. See, for instance (Chuah, 2012, §2). Let θ, σ be $\mathfrak{g}$-involutions. They correspond to some real forms $\mathfrak{g}_\mathbb{R}$, $\mathfrak{g}'_\mathbb{R}$ of $\mathfrak{g}$. If $\mathfrak{g}^\theta \cong \mathfrak{g}^\sigma$, then $\mathfrak{g}_\mathbb{R} \cong \mathfrak{g}'_\mathbb{R}$ (Helgason, 2001, Chapter X-6, Theorem 6.2) and hence θ, σ are equivalent. This proves Theorem 1.3(a) for involutions.

Next, we prove Theorem 1.3(a) for $\mathfrak{g}$-automorphisms θ, σ of order 3. Theorems 1.1 and 1.2 are represented by Kac diagrams *c, c*' in Figures 1, 2, or 3. Suppose that θ, σ are not equivalent to each other. By Theorem 2.1, *c, c*' are not equivalent. By Proposition 3.1, ker *c* $\ncong$ ker *c* '. By Theorem 2.1, ker *c*, ker *c*' are the Dynkin diagrams of the semisimple parts of $\mathfrak{g}^\theta$, $\mathfrak{g}^\sigma$, so we have $\mathfrak{g}^\theta \ncong \mathfrak{g}^\sigma$. This proves Theorem 1.3(a) for automorphisms of order 3.

Next, we prove Theorem 1.3(b). Let $n \geq 4$. It suffices to construct counterexamples of non-equivalent $\mathfrak{g}$-automorphisms θ, σ of order *n* such that $\mathfrak{g}^\theta \cong \mathfrak{g}^\sigma$. Let $\mathfrak{g} = A_m$, where $m \geq 3$. Consider the two Kac diagrams in Figure 4. They are not equivalent, but their kernels are the same, namely the Dynkin diagram of A_{m-2}.

Suppose that θ, σ is represented by the Kac diagrams in Figure 4. By Theorem 2.1, θ and σ are not equivalent because their Kac diagrams are not equivalent, but both their invariant subalgebras are $\mathfrak{g}^\theta = \mathfrak{g}^\sigma = A_{m-2} + \mathbb{C}^2$. With this counterexample, we have proved Theorem 1.3(b).

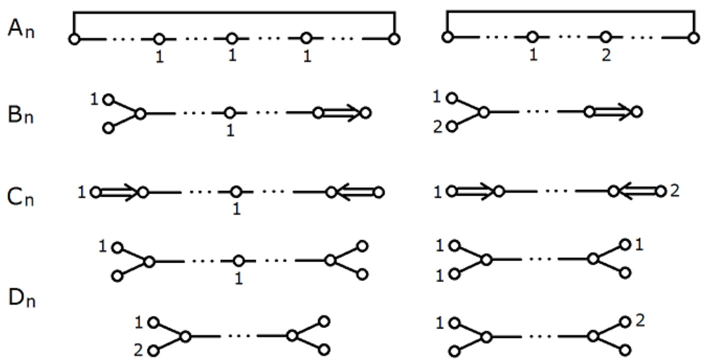

Fig. 1. Inner classical Kac diagrams of order 3

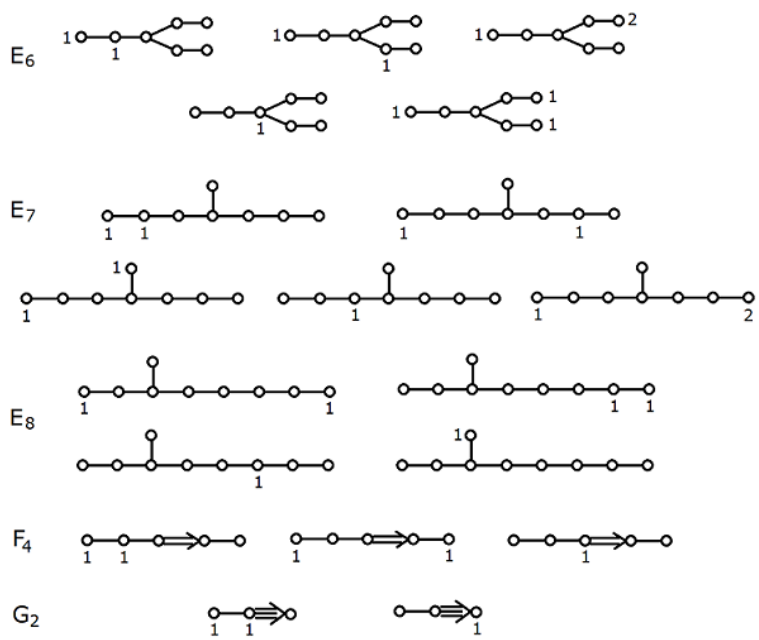

Fig. 2. Exceptional Kac diagrams of order 3.

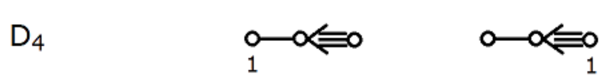

Fig. 3. Outer Kac diagrams of order 3.

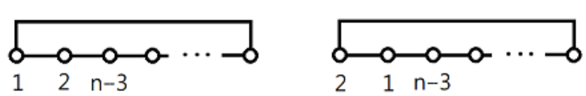

Fig. 4. Non-equivalent Kac diagrams with the same kernel.

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