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Boehmian spaces for a class of Whittaker integral transformations

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Abstract

In this paper, we are concerned with a class of Whittaker integral transforms involving confluent hypergeometric functions as kernels. We present certain convolution products and derive the prerequisite axioms for generating the relevant spaces of Boehmians. We further give the definition and discuss the inverse problem of the Whittaker integral in a generalized sense. Moreover, the integral in question has been shown to be linear and consistent with the classical one. Certain results are also discussed in some details.

Keywords: Boehmian space; Laplace transform; Mellin transform; Whittaker transform; Whittaker function.

AMS Subject Classification: 44A15; 44A35.

1. Introduction

There has been a great revival of interest in the study of hypergeometric functions and integrals involved with them in the last two decades. This interest mainly comes from the connection between hypergeometric functions and many areas of mathematics such as representation theory, algebraic geometry, Hodge theory, combinatorics, D-modules, number theory and mirror symmetry as well.

As one of generalizations of the classical Laplace integral transforms

$$Lf(x)=\int_0^\infty e^{-xt}\varphi(t)\mathrm{d}t, \tag{1.1}$$

the generalized Whittaker transform associated with geometric functions was introduced by Srivastava (1968) and studied further by Tiwari & Aye (1982) as

$$\mathbf{w}_{k,m}^{(\sigma)}\varphi(x)=\int_0^\infty (xt)^{\sigma-(1/2)}\exp(-(1/2)qxt)\mathbf{w}_{k,m}(pxt)\varphi(t)\mathrm{d}t, \tag{1.2}$$

where $x>0$ and $\mathbf{w}_{k,m}$ represents a Whittaker function (Slater, 1960)

$$\mathbf{w}_{k,m}(\omega)=e^{-\frac{1}{2}z}\sum_{m,-m}\frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-k-m\right)}\omega^{m+\frac{1}{2}}{}_1F_1\left[\frac{1}{2}-k+m;1+2m;\omega\right], \tag{1.3}$$

defined in terms of the Kummer's confluent hypergermetric function

$${}_1F_1[a;b;\omega]=\sum_{n=0}^\infty\frac{(a)_n}{(b)_n}\frac{\omega^n}{n!}. \tag{1.4}$$

As a special case where $\sigma=m$, $k+m=\frac{1}{2}$ and $p=q=1$, the Whittaker integral (1.2) is indeed gives the Laplace integral (1.1).

Following inversion theorem is due to Srivastava (1968).

Theorem 1.1. If $\mathbf{w}_{k,m}^{(\sigma)}\varphi$ is the Whittaker integral transform of φ. Then we have

$$\varphi(t)=\int_0^\infty \Phi(xt)\mathbf{w}_{k,m}^{(\sigma)}\varphi(x)\mathrm{d}x, \tag{1.5}$$

where

$$\Phi(x)=\frac{1}{2\pi i}\int_{c+i\infty}^{c+i\infty}\frac{x^{-\omega}}{\Psi(1-\omega)}\mathrm{d}\omega$$

and

$$\Psi(\omega)=\frac{p^{m+(1/2)}\Gamma(\sigma\mp m+\omega)}{\left(\frac{1}{2}(q+p)\right)^{\sigma+m+\omega}\Gamma\left(\sigma-k+\omega+\frac{1}{2}\right)}{}_2F_1\left[\begin{matrix}\sigma+m+\omega,m-k+\frac{1}{2};\ \frac{q-p}{q+p}\\ \sigma-k+\omega+\frac{1}{2};\end{matrix}\right].$$

Provided the integral equation $\Phi(x)$ exists, The equation (1.5) is convergent and it further satisfies

$$x^{-c}\mathbf{w}_{k,m}^{(\sigma)}\varphi(x)\in\mathbf{l}(0,\infty) \text{ and } x^{c-1}\varphi(x)\in\mathbf{l}(0,R_0),$$

for $R_0>0$ and $Re(\sigma\mp m+1)>c>0$.

Following known property is due to Whittaker & Watson (1946)

$$\mathbf{w}_{k,m}(\omega)=\begin{cases} O\left(\omega^k e^{-(\omega/2)}\right), & |\omega|\to\infty \\ O\left(\omega^{(1/2)+m}\right), & |\omega|\to 0 \\ O\left(\omega^{(1/2)-m}\right), & |\omega|\to 0 \end{cases}$$

where $m<0$.

Further in this theory and, in agreement to that of Srivastava (1968), a modification of the Laplace integral (1.1) was earlier proposed by Varma (1951) in the form

$$\mathbf{v}_{k,m}\varphi(x)=\int_0^\infty (xt)^{m-\frac{1}{2}}e^{-\frac{1}{2}xt}\mathbf{w}_{k,m}(xt)\varphi(t)\mathrm{d}t, \quad (1.6)$$

where $\mathbf{w}_{k,m}$ ia a Whittaker function.

2. Construction of Boehmian spaces

In this section we assume readers are aquainted with the abstract construction of Boehmian spaces. If it were otherwise, we refer to Al-omari (2013); Nemzer (2009); Al-omari & Al-omari (2015); Mikusinski & Zayed (1993) and Al-omari & Agarwal (2015) and many others, to mention but a few. Let $\boldsymbol{\kappa}$ be the Schwartz' space of test functions of bounded supports in $(0,\infty)$; i.e., an integrable function φ is said to be in $\boldsymbol{\kappa}$ if it is smooth and zero outside some finite interval.

On the otherhand, a function $\varphi(t)$, defined on $(0,\infty)$, is said to be a member of the space $\mathbf{m}_{c,d,\alpha}$ if it is smooth and it satisfies the set of seminorms

$$\mathbf{i}_{c,d,\ell}(\varphi):=\sup_{t\in(0,\infty)}\left|\boldsymbol{\lambda}_{c,d}(t)(-tD_t)^\ell(t\varphi(t))\right|\le C_\ell A^\ell \ell^{\ell\alpha}$$

where

$$\boldsymbol{\lambda}_{c,d}(t)=\begin{cases} t^{-c}, & 0<t\le 1 \\ t^{-d}, & 1<t<\infty \end{cases}, \quad (2.1)$$

C_ℓ, A^ℓ are constants depending on $\varphi, \alpha\ge 0, \ell=0,1,2,...$.

Denote by $\mathbf{m}_{c,d}$ the union space

$$\mathbf{m}_{c,d}=\bigcup_{i=1}^\infty \mathbf{m}_{c,d,\alpha i}. \quad (2.2)$$

Then, $\varphi\in\mathbf{m}_{c,d}$ provided φ is smooth and that

$$\|\varphi\|_{c,d,\ell}:=\mathbf{i}_{c,d,\ell}(\varphi):=\sup_{t\in(0,\infty)}\left|\boldsymbol{\lambda}_{c,d}(t)(-tD_t)^\ell(t\varphi(t))\right|<\infty, \quad (2.3)$$

where $D_t=\frac{d}{dt}$ is a differential operator.

In the literature, it is has been shown that $\boldsymbol{\kappa}(0,\infty)\subset\mathbf{m}_{c,d,\alpha}$ and, hence, $\mathbf{m}'_{c,d,\alpha}\subset\boldsymbol{\kappa}'(0,\infty)$, where $\mathbf{m}'_{c,d,\alpha}$ and $\boldsymbol{\kappa}'(0,\infty)$ are the dual spaces of $\mathbf{m}_{c,d,\alpha}$ and $\boldsymbol{\kappa}(0,\infty)$, respectively.

Further, due to Tiwari & Aye (1982), the kernel function

$$(xt)^{\sigma-(1/2)}\exp(-(1/2)qxt)\mathbf{w}_{k,m}(pxt)$$

of $\mathbf{w}_{k,m}^{(\sigma)}$ transform is a member of $\mathbf{m}_{c,d}$ provided that $Re\{(q+p)s\}>2c, m>0$ and $Re(\sigma-m)\ge 0$.

The Mellin-type convolution product of two integrable functions φ and g is defined by Zemanian (1987) as

$$(\varphi\times g)(x)=\int_0^\infty y^{-1}\varphi(xy^{-1})g(y)\mathrm{d}y \ (x>0), \quad (2.4)$$

provided the integral exists.

Properties of (2.4) can be enumerated as follows:

$$\left.\begin{array}{l} (\mathrm{i}): \varphi\times g(x)=g\times\varphi(x); \\ (\mathrm{ii}): \varphi\times(g+h)=\varphi\times g(x)+\varphi\times h(x); \\ (\mathrm{iii}): (\alpha\varphi)\times g(x)=\alpha(\varphi\times g)(x); \\ (\mathrm{v}): (\varphi\times g)\times h(x)=\varphi\times(g\times h)(x). \end{array}\right\}. \quad (2.5)$$

For more details of the Mellin-type convolution (2.4), we refer readers to the cited book of Dimovski (1982,1990).

Now we present the convolution product that we need in our next investigation.

Definition 2.1. Let φ_1 and φ_2 be integrable functions defined on $(0,\infty)$. Between φ_1 and φ_2 we define an operation $\otimes$ by the integral equation

$$\varphi_1\otimes\varphi_2(x)=\int_0^\infty \varphi_1(xy)\varphi_2(y)\mathrm{d}y \ (x>0) \quad (2.6)$$

when the integral exists.

Let us now derive some axioms that legitimate the existence of the Boehmian space $\beta(\mathbf{m}_{c,d},(\boldsymbol{\kappa},\times),\Delta,\otimes)$ where Δ is a collection of delta sequences from $\boldsymbol{\kappa}(0,\infty)$ such that

$$(\mathrm{i})\int_0^\infty \delta_n(x)\mathrm{d}x=1(\forall n\in\mathbb{N}); \quad (2.7)$$

$$(\mathrm{ii})|\delta_n(x)|\le M_1(\forall n\in\mathbb{N}); \quad (2.8)$$

$$(\mathrm{iii})\, supp\,\delta_n(x)\subset(a_n,b_n), \quad (2.9)$$

where M_1 is a positive constant and a_n, $b_n \to 0$ as $n \to \infty$.

Theorem 2.2. Let $\varphi \in \mathbf{m}_{c,d}$ and $\phi \in \boldsymbol{\kappa}(0,\infty)$. Then we have

$$\varphi \otimes \phi \in \mathbf{m}_{c,d}.$$

Proof. Let $\varphi \in \mathbf{m}_{c,d}$. Then $\mathbf{i}_{c,d,\ell}(\varphi) < M$, $0 < M \in \mathbb{R}$. The hypothesis that $\phi \in \boldsymbol{\kappa}(0,\infty)$ implies $supp\phi \subseteq [a,b]$, for some real numbers $a, b \in \mathbb{R}$, $0 < a < b$. Hence, this reveals

$$\mathbf{i}_{c,d,\ell}(\varphi \otimes \phi)(t) = \sup\left|\boldsymbol{\lambda}_{c,d}(t)(-tD_t)^{\ell}\left(t\int_0^{\infty} \varphi(tx)\phi(x)\mathrm{d}x\right)\right|$$
$$= \mathbf{i}_{c,d,\ell}(\varphi)(t)\int_a^b |\phi(x)|\mathrm{d}x.$$

Since $\phi \in \boldsymbol{\kappa}(0,\infty)$, it can be deduced that

$$\mathbf{i}_{c,d,\ell}(\varphi \otimes \phi)(t) \le M_2 < \infty,$$

where

$$M_2 = M\int_a^b |\phi(x)|\mathrm{d}x.$$

This completes the proof of the theorem.

Theorem 2.3. Let $\varphi \in \mathbf{m}_{c,d}$ and $\phi_1, \phi_2 \in \boldsymbol{\kappa}(0,\infty)$. Then we have

$$\varphi \otimes (\phi_1 \times \phi_2)(x) = (\varphi \otimes \phi_1) \otimes \phi_2.$$

Proof. Let the hypothesis of the theorem be satisfied. Then using (2.4), (2.6) and Fubini's theorem imply

$$\varphi \otimes (\phi_1 \times \phi_2)(x) = \int_0^{\infty} \varphi(xy)\phi_1 \times \phi_2(y)\mathrm{d}y$$
$$= \int_0^{\infty} t^{-1}\phi_2(t)\int_0^{8} \varphi(xy)\phi_1(yt^{-1})\mathrm{d}y\mathrm{d}t. \qquad (2.10)$$

Change of variables $(zt = y)$ on (2.10) implies

$$\varphi \otimes (\phi_1 \times \phi_2)(x) = \int_0^{\infty} \phi_2(t)\int_0^{\infty} \varphi(xzt)\phi_1(z)\mathrm{d}z\mathrm{d}t$$
$$= \int_0^{\infty} (\varphi \otimes \phi_1)(xt)\phi_2(t)\mathrm{d}t.$$

This completes the proof of the theorem.

Theorem 2.4. Let $(\delta_n), (\varepsilon_n) \in \Delta$. Then $\delta_n \times \varepsilon_n \in \Delta$.

Proof. of this theorem follows from the properties of Δ sequences, (2.7)-(2.9). We prefer we omit the details.

Following theorem is straightforward from usual properties of integration.

Theorem 2.5. Let $(\varphi_n), \varphi \in \mathbf{m}_{c,d}$ and $\phi_1, \phi_2, \phi \in \boldsymbol{\kappa}(0,\infty)$. Then the following satisfy.

(i) if $\varphi_n \to \varphi$ in $\mathbf{m}_{c,d}$, then $\varphi_n \otimes \phi \to \varphi \otimes \phi$ as $n \to \infty$,

(ii) $\varphi_n \otimes (\phi_1 + \phi_2) = \varphi_n \otimes \phi_1 + \varphi_n \otimes \phi_2$,

(iii) $\alpha(\varphi_n \otimes \phi) = (\alpha\varphi_n) \otimes \phi$, $\alpha \in \mathbb{C}$.

To get our space established it is sufficient to establish the following theorem.

Theorem 2.6. Let $(\delta_n) \in \Delta$ and $\varphi \in \mathbf{m}_{c,d}$. Then $\varphi \otimes \delta_n \to \varphi$ as $n \to \infty$.

Proof. By invoking (2.7) of delta sequences, we get

$$\mathbf{i}_{c,d,\ell}(\varphi \otimes \phi - \varphi)(t) = \sup_{t\in(0,\infty)}\left|\boldsymbol{\lambda}_{c,d}(t)(-tD_t)^{\ell}(t(\varphi \otimes \delta_n - \varphi)(t))\right|$$

i.e. $$= \sup_{t\in(0,\infty)}\left|\boldsymbol{\lambda}_{c,d}(t)(-tD_t)^{\ell}\left(t\left(\int_0^{\infty} (\varphi(ty) - \varphi(y))\delta_n(y)\mathrm{d}y\right)\right)\right|$$

i.e. $$= \sup_{t\in(0,\infty)}\int_0^{\infty} \left|\boldsymbol{\lambda}_{c,d}(t)(-tD_t)^{\ell}(t(\varphi(ty) - \varphi(y)))\right||\delta_n(y)|\mathrm{d}y.$$

By aid of (2.8) and (2.9) we obtain

$$\mathbf{i}_{c,d,\ell}(\varphi \otimes \phi - \varphi)(t) \le M_1(b_n - a_n)\sup_{t\in(0,\infty)}\left|\boldsymbol{\lambda}_{c,d}(t)(-tD_t)^{\ell}(t(\varphi(ty) - \varphi(y)))\right|. \qquad (2.11)$$

Once again, on account of (2.9) and the property that $\varphi \in \mathbf{m}_{c,d}$; (2.11) gives

$$\mathbf{i}_{c,d,\ell}(\varphi \otimes \phi - \varphi)(t) \to 0 \text{ as } n \to \infty.$$

Therefore $\varphi \otimes \delta_n \to \varphi$ as $n \to \infty$ in $\mathbf{m}_{c,d}$.

Hence the theorem is completely proved.

The Space $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$ is performed. The sum, multiplication by a scalar, the operation $\otimes$ and differentiation of Boehmians can be respectively defined in a natural way

(i) $\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right] + \left[\frac{\{g_n\}}{\{\tau_n\}}\right] = \left[\frac{\{\varphi_n \otimes \tau_n\} + \{g_n \otimes \varepsilon_n\}}{\{\varepsilon_n \times \tau_n\}}\right]$,

(ii) $\lambda\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right] = \left[\frac{\{\lambda\varphi_n\}}{\{\varepsilon_n\}}\right]$, λ being complex number,

(iii) $\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right] \otimes \left[\frac{\{g_n\}}{\{\tau_n\}}\right] = \left[\frac{\{\varphi_n \otimes g_n\}}{\{\varepsilon_n \times \tau_n\}}\right]$,

(iv) $\mathrm{D}^{\alpha}\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right] = \left[\frac{\{\mathrm{D}^{\alpha}\varphi_n\}}{\{\varepsilon_n\}}\right]$.

The operation $\otimes$ is extended to $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)\times\boldsymbol{\kappa}$ by

$$\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right]\otimes\phi=\left[\frac{\{\varphi_n\otimes\phi\}}{\{\varepsilon_n\}}\right].$$

Construction of the space $\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$ can be obtained by the properties (2.5) of $\times$ and analysis similar to that used for the space $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$. We, therefore, omit the similar proofs.

In that manner, sum, multiplication by a scalar, the operation $\times$ and differentiation of Boehmians can respectively be defined as

(i) $\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right]+\left[\frac{\{g_n\}}{\{\tau_n\}}\right]=\left[\frac{\{\varphi_n\times\tau_n\}+\{g_n\times\varepsilon_n\}}{\{\varepsilon_n\times\tau_n\}}\right]$,

(ii) $\lambda\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right]=\left[\frac{\{\lambda\varphi_n\}}{\{\varepsilon_n\}}\right]$, λ being complex number,

(iii) $\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right]\times\left[\frac{\{g_n\}}{\{\tau_n\}}\right]=\left[\frac{\{\varphi_n\times g_n\}}{\{\varepsilon_n\times\tau_n\}}\right]$,

(iv) $\mathsf{D}^\alpha\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right]=\left[\frac{\{\mathsf{D}^\alpha\varphi_n\}}{\{\varepsilon_n\}}\right]$.

The operation $\times$ is extended to $\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)\times\boldsymbol{\kappa}$ by $\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right]\times\phi=\left[\frac{\{\varphi_n\times\phi\}}{\{\varepsilon_n\}}\right]$.

3. Generalization of Whittaker transform

To define the Whittaker integral transform of a Boehmian in $\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$, we establish the following theorem.

Theorem 3.1. Let $\{\varphi_n\}$ and $\{\delta_n\}$ belong to $\mathbf{m}_{c,d}$ and Δ, respectively. Then we have

$$\mathbf{w}_{k,m}^{(\sigma)}(\varphi_n\times\delta_n)(x)=\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\otimes\delta_n(x).$$

Proof. Let $\{\varphi_n\}\in\mathbf{m}_{c,d}$ and $(\delta_n)\in\Delta$ be given. Then with aid of the integral equation (1.2) and the convolution product (2.4) we write

$$\mathbf{w}_{k,m}^{(\sigma)}(\varphi_n\times\delta_n)(x)=\int_0^\infty\delta_n(y)y^{-1}\int_0^\infty(xt)^{\sigma-\frac{1}{2}}\exp\left(-\frac{1}{2}qxt\right)$$
$$\times\mathbf{w}_{k,m}(pxt)\varphi_n(ty^{-1})\mathrm{d}t\mathrm{d}y. \qquad (3.1)$$

By change of variables on (3.1) we obatin

$$\mathbf{w}_{k,m}^{(\sigma)}(\varphi_n\times\delta_n)(x)=\int_0^\infty\delta_n(y)\mathbf{w}_{k,m}^{(\sigma)}\varphi_n(yx)\mathrm{d}y.$$

Hence, (2.6) yields

$$\mathbf{w}_{k,m}^{(\sigma)}(\varphi_n\times\delta_n)(x)=(\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\otimes\delta_n)(x).$$

This completes the proof of the theorem.

In view of Theorem 3.1, we are lead to the following definition.

Definition 3.2. Let $\beta\in\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$ where $\beta=\left[\frac{\{\varphi_n\}}{\{\varepsilon_n\}}\right]$. Then a generalization of $\mathbf{w}_{k,m}^{(\sigma)}$ is defined as

$$\overset{\circ}{\mathbf{w}}{}_{k,m}^{(\sigma)}\beta=\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\varepsilon_n\}}\right] \qquad (3.2)$$

in $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$.

Theorem 3.3. $\overset{\circ}{\mathbf{w}}{}_{k,m}^{(\sigma)}:\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)\to\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$ is well-defined.

Proof. Let $\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right]=\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\in\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$. Then by the concept of quotients of $\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$ we get

$$\varphi_n\times\varepsilon_m=g_m\times\delta_n(m,n\in\mathsf{N}).$$

Hence,

$$\mathbf{w}_{k,m}^{(\sigma)}(\varphi_n\times\varepsilon_m)=\mathbf{w}_{k,m}^{(\sigma)}(g_m\times\delta_n)(m,n\in\mathsf{N}).$$

Therefore, Theorem 3.1 gives

$$\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\otimes\varepsilon_m=\mathbf{w}_{k,m}^{(\sigma)}v_m\otimes\delta_n(m,n\in\mathsf{N}).$$

Concept of quotients and equivalent classes of $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$ imply

$$\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right]=\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}g_n\}}{\{\varepsilon_n\}}\right](m,n\in\mathsf{N}).$$

This completes the proof of the theorem.

Theorem 3.4. The operator $\overset{\circ}{\mathbf{w}}{}_{k,m}^{(\sigma)}:\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)\to\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$ is linear.

Proof. Let $\beta_1,\beta_2\in\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$ be given as $\beta_1=\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right],\beta_2=\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]$. Then

$$\beta_1+\beta_2=\left[\frac{\{\varphi_n\times\varepsilon_n+g_n\times\delta_n\}}{\{\delta_n\times\varepsilon_n\}}\right]. \qquad (3.3)$$

By addition of Boehmians of $\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$, Theorem 3.1 and linearity of the Whittaker transform, (2.11) becomes

$$\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta_1+\beta_2)=\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}(\varphi_n\times\varepsilon_n+g_n\times\delta_n)\}}{\{\delta_n\times\varepsilon_n\}}\right]=\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\otimes\varepsilon_n+\mathbf{w}_{k,m}^{(\sigma)}g_n\otimes\delta_n\}}{\{\delta_n\times\varepsilon_n\}}\right].$$

Hence, addition of Boehmians of $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$ leads to

$$\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta_1+\beta_2)=\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right]+\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}g_n\}}{\{\varepsilon_n\}}\right]=\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\,\beta_1+\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\,\beta_2.$$

Moreover, if $\alpha^*\in C$; then it easy to see that $\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\alpha^*\beta_1)=\alpha^*\,\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta_1)$.

This completes the proof of the theorem.

Theorem 3.5. Let $\beta\in\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times),\beta=0$. Then $\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\,\beta=0$.

Proof of this theorem is straightforward. Details are, therefore, omitted.

Theorem 3.6. Let $\beta_0,\beta\in\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$. Then $\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta_0\times\beta)=\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta_1)\otimes\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta_2)$ in $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$.

Proof. Let $\beta_0,\beta\in\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$ be given such that $\beta_0=\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right]$ and $\beta=\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]$. Then, by applying $\times$ for $\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$ we get

$$\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta_0\times\beta_2)=\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\left(\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right]\times\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right]\right)=\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\left(\left[\frac{\{\varphi_n\times g_n\}}{\{\delta_n\times\varepsilon_n\}}\right]\right).$$

Hence, by Theorem 3.1 and the operation $\otimes$ in $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$ we write

$$\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta_0\times\beta)=\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}(\varphi_n\times g_n)\}}{\{\delta_n\times\varepsilon_n\}}\right]=\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right]\otimes\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}g_n\}}{\{\varepsilon_n\}}\right].$$

Hence, it follows that $\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta_0\times\beta)=\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta_0)\otimes\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}(\beta)$ in $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$.

The theorem is proved.

Definition 3.7. Let $\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right]\in\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$. Then we define the inverse transform of $\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right]$ as

$$\left(\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\right)^{-1}\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right]=\left[\frac{\left(\mathbf{w}_{k,m}^{(\sigma)}\right)^{-1}\left(\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\right)}{\{\delta_n\}}\right]=\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right]\quad(3.4)$$

for each $\{\delta_n\}\in\Delta$.

Theorem 3.8. $\left(\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\right)^{-1}:\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)\to\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$ is well-defined and linear.

Proof. Let $\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right]=\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}g_n\}}{\{\varepsilon_n\}}\right]$ in $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$.

Then, we have $\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\otimes\varepsilon_m=\mathbf{w}_{k,m}^{(\sigma)}g_m\otimes\delta_n$. By Theorem 3.1. we have $\mathbf{w}_{k,m}^{(\sigma)}(\varphi_n\times\varepsilon_m)=\mathbf{w}_{k,m}^{(\sigma)}(g_m\times\delta_n)$.

Therefore, $\varphi_n\times\varepsilon_m=g_m\times\delta_n$. That is, the quotients $\frac{\{\varphi_n\}}{\{\delta_n\}},\frac{\{g_n\}}{\{\varepsilon_n\}}$ are equivalent in $\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$. Hence, by (3.1), we have

$$\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right]=\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right].$$

Hence, $\left(\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\right)^{-1}$ is well-defined operator.

We show that $\left(\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\right)^{-1}$ is linear.

Let $\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right],\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}g_n\}}{\{\varepsilon_n\}}\right]$ in $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$.

Then addition defined in $\beta_2(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\otimes)$ and Theorem 3.1 reveal

$$\left(\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\right)^{-1}\left(\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right]+\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}g_n\}}{\{\varepsilon_n\}}\right]\right)=$$

$$\left(\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\right)^{-1}\left(\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}(\varphi_n\otimes\varepsilon_n)+\mathbf{w}_{k,m}^{(\sigma)}(g_n\otimes\delta_n)\}}{\{\delta_n\}\times\{\varepsilon_n\}}\right]\right).$$

Therefore, (3.4) implies

$$\left(\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\right)^{-1}\left(\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right]+\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}g_n\}}{\{\varepsilon_n\}}\right]\right)=\left[\frac{\{\varphi_n\otimes\varepsilon_n+g_n\otimes\delta_n\}}{\{\delta_n\}\times\{\varepsilon_n\}}\right].$$

Addition of $\beta_1(\mathbf{m}_{c,d};(\boldsymbol{\kappa},\times);\Delta;\times)$ also gives

$$\left(\overset{\circ\,(\sigma)}{\mathbf{w}}_{k,m}\right)^{-1}\left(\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}\varphi_n\}}{\{\delta_n\}}\right]+\left[\frac{\{\mathbf{w}_{k,m}^{(\sigma)}g_n\}}{\{\varepsilon_n\}}\right]\right)=\left[\frac{\{\varphi_n\}}{\{\delta_n\}}\right]\left[\frac{\{g_n\}}{\{\varepsilon_n\}}\right].$$

This completes the proof of the theorem

Conclusion

Certain convolution products and two classes of Boehmians have been derived. The Whittaker transform whose kernel involving confluent hypergeometric functions has therefore been extended to a context of Boehmians. Over and above, the theory and properties of the considered integral and its inverse have been duly adapted to coincide with the existed theory.

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فضاءات بوهيمية لصف من تحويلات ويتاكر التكاملية

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خلاصة

نهتم في هذا البحث بأحد أصناف تحويلات ويتاكر التكاملية التي نواقها دوال مندمجة فوهندسية. و نقوم بعرض بعض التلافية و نستخرج الموضوعات المسبقة لتوليد الفضاءات البوهيمية المناسبة. كما نقوم بإعطاء التعريف و مناقشة المسألة العكسية لتكامل ويتاكر بشكل أكثر تعميماً. بالإضافة إلى ذلك، نثبت أن التكاملات موضوع النقاش هي خطية و متسقة مع التكاملات الكلاسيكية . كما نجري نقاشاً لبعض نتائجنا.