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Some integral operators and their properties

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Abstract

In this study, some integral operators, which have broad applications in the theory of elementary particles and scattering, have been investigated in Holder space. We show that some important inequalities for the norm of these operators are also satisfied in Holder space.

Keywords: Cauchy integral operator; integral operator.

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1. Introduction

It is well known that the theory of singular integral equations has broad applications in many theoretical and practical investigations of problems in mathematics, mathematical physics, hydrodynamics and elasticity theory (Kalandia, 1973; Parton & Perlin, 1977; Colton & Kress, 1983; Duduchava, 1982; Panasyuk *et al.,* 1984; Reşidoğlu, 2001).

Also, on the theoretical analysis of singular integral equations and on the approximate solution, there exist a lot of works (Ivanov, 1968; Lu, 1993; Mustafa, 2008; Mustafa & Khalilov, 2009; Mustafa & Caglar, 2010; Mustafa, 2013).

Consider the following nonlinear singular integral equation

$$\varphi(t) = f(t)\left\{\varphi^2(t) + \left[\lambda - S\varphi(t) + \mu S_+\varphi(t)\right]^2\right\},\ t \in [0,1]. \quad (1.1)$$

Here,

$$S\varphi(t) = \frac{1}{\pi}\int_0^1 \frac{\varphi(\tau)}{\tau - t}d\tau \text{ and } S_+\varphi(t) = \frac{1}{\pi}\int_0^1 \frac{\varphi(\tau)}{\tau + t}d\tau. \quad (1.2)$$

Nonlinear singular integral equation (1.1) has crucial applications in the theory of elementary particles and scattering (Colton & Kress, 1983). It is important to examine the type of such equations. In the investigation of the existence of the solution of equation (1.1) it is important to examine the operators (1.2).

In this paper, we prove that the operators S and S_+ from (1.2) are bounded in Holder space. Moreover, for these operators, some important inequalities in the different norms are also given.

2. Preliminaries

In this section, we will introduce some necessary information required for the proof of main results.

As usual, throughout the work, $C[0,1]$ is the set of continuous functions defined on $[0,1]$ with maximum norm

$$\|f\|_\infty = \max\left\{|f(t)| : t \in [0,1]\right\}.$$

Definition 2.1. (Daugavet, 1977). The function

$$\omega(\varphi, x) = \sup\left\{|\varphi(t_2) - \varphi(t_1)| : |t_2 - t_1| \le x\right\}, x \in [0, +\infty)$$

is called the modulus of continuity of the bounded function $\varphi : \mathbb{R} \to \mathbb{R}$.

Let us recall the properties of the modulus of continuity:

1. The modulus of continuity is a continuous function.
2. The modulus of continuity is a non-decreasing function.
3. For every $x_1, x_2 > 0$, $\omega(x_1 + x_2) \le \omega(x_1) + \omega(x_2)$.
4. $\omega(0) = 0$.
5. For every $x > 0$ and $\alpha > 0$ $\frac{\omega(x)}{x}$ and $\frac{\omega(x)}{x^\alpha}$ are decreasing functions.

Definition 2.2. (Lu, 1993). Assume $\varphi(t)$ is defined on $[0,1]$. If

$$\left|\varphi(x_2)-\varphi(x_1)\right| \le K\left|x_2-x_1\right|^{\alpha}, 0<\alpha\le 1$$

for arbitrary points $x_1, x_2 \in [0,1]$, where $K>0$ and α are definite constants, then $\varphi(t)$ is said to satisfy the Holder condition of order α, or simply condition H_α, denoted by $\varphi \in H_\alpha[0,1]$.

The functions in class H possess the following properties:

1. If $\varphi \in H[0,1]$, then $\varphi \in C[0,1]$, i.e., $H[0,1] \subset C[0,1]$.
2. If $\varphi \in H_\alpha[0,1]$ and $0<\beta\le\alpha$, then $\varphi \in H_\beta[0,1]$ i.e., $H_\alpha[0,1] \subseteq H_\beta[0,1]$, if $0<\beta\le\alpha$.
3. If both φ and $\psi \in H_\alpha[0,1]$, then so do $\varphi \pm \psi, \varphi\cdot\psi, \varphi/\psi$ ($\psi \ne 0$ on $[0,1]$).
4. If both φ and $\psi \in H_\alpha[0,1]$, then so do $\varphi+\psi, \varphi\cdot\psi$ and

$$H(\varphi+\psi;\alpha)=H(\varphi;\alpha)+H(\psi;\alpha)$$
$$H(\varphi\cdot\psi;\alpha)\le H(\varphi;\alpha)\|\psi\|_\infty+\|\varphi\|_\infty H(\psi;\alpha).$$

Let $\overset{\circ}{C}[0,1]=\{f \in C[0,1]: f(0)=0=f(1)\}$ and $\overset{\circ}{H}_\alpha[0,1]=\{\varphi\in H_\alpha:\varphi(0)=0=\varphi(1)\}$.

The function spaces $H_\alpha[0,1]$ and $\overset{\circ}{H}_\alpha[0,1]$ are Banach spaces with norm

$$\|\varphi\|_\alpha=\max(\|\varphi\|_\infty, H(\varphi;\alpha)).$$

Here,

$$H(\varphi;\alpha)=\sup\left\{\frac{\omega(\varphi,x)}{x^\alpha}:0<x\le 1\right\}.$$

Furthermore, throughout the paper, we denote H_α ($\overset{\circ}{H}_\alpha$) instead of $H_\alpha[0,1]$ ($\overset{\circ}{H}_\alpha[0,1]$), unless stated otherwise.

We denote the norm of function $\varphi\in\overset{\circ}{H}_\alpha$ by

$$\|\varphi\|_{\alpha,0}=\max(\|\varphi\|_\infty, H(\varphi;\alpha)).$$

The norm of a bounded linear operator $\Im:\overset{\circ}{H}_\alpha\to H_\alpha$ is defined as follows (Kreyszig, 1978):

$$\|\Im\|_\alpha=\|\Im\|_{\overset{\circ}{H}_\alpha\to H_\alpha}=\sup_{\varphi\ne 0}\left\{\frac{\|\Im\varphi\|_\alpha}{\|\varphi\|_{\alpha,0}}:\varphi\in\overset{\circ}{H}_\alpha\right\}$$

$$\|\Im\|_\infty=\|\Im\|_{\overset{\circ}{H}_\alpha\to C[0,1]}=\sup_{\varphi\ne 0}\left\{\frac{\|\Im\varphi\|_\infty}{\|\varphi\|_{\alpha,0}}:\varphi\in\overset{\circ}{H}_\alpha,\right\}.$$

Let

$$J_0=\left\{\varphi\in C[0,1]:\int_0^1\frac{\omega(\varphi,\xi)}{\xi}<+\infty\right\}$$

and

$$Z(\omega(\varphi,\cdot),t)=\int_0^t\frac{\omega(\varphi,\xi)}{\xi}d\xi+t\int_t^1\frac{\omega(\varphi,\xi)}{\xi^2}d\xi, t\in[0,1].$$

Then, the following is provided:

1. If $\varphi\in\overset{\circ}{H}_\alpha$, then $\varphi\in J_0$, i.e., $\overset{\circ}{H}_\alpha\subset J_0$.
2. $Z(\omega(\varphi,\cdot),t)$ is a non-decreasing function on $[0,1]$.

3. Main results

In this section of the paper, we provide some properties of the operators $S:\overset{\circ}{H}_\alpha\to H_\alpha$ and $S_+:\overset{\circ}{H}_\alpha\to H_\alpha$, which are defined in formula (1.2).

Theorem 3.1. Let the operators $S:\overset{\circ}{H}_\alpha\to H_\alpha$ and $S_+:\overset{\circ}{H}_\alpha\to H_\alpha$ be defined as in formula (1.2) and $\varphi\in J_0$. Then, for every $x\in(0,1]$,

$$\omega(S\varphi,x)\le c_1 Z(\omega(\varphi,\cdot),x), \quad (3.1)$$

$$\omega(S_+\varphi,x)\le c_2 Z(\omega(\varphi,\cdot),x). \quad (3.2)$$

Here, $c_1=\frac{1}{\pi}(\frac{67}{6}+\ln 3)$, $c_2=\frac{2}{\pi}$ are definite constants.

Proof. Let

$$\phi(t)=\begin{cases}\varphi(t) & \text{for } t\in[0,1],\\ 0 & \text{for } t\in[-1,2]/[0,1]\end{cases}$$

and

$$F\phi(t)=\frac{1}{\pi}\int_{-1}^{2}\frac{\phi(\tau)}{\tau-t}d\tau,\ t\in(-1,2).$$

The operator $F:\overset{\circ}{H}_\alpha\to H_\alpha$ can be written as

$$F\phi(t)=\frac{1}{\pi}\int_{-1}^{2}\frac{\phi(\tau)-\phi(t)}{\tau-t}d\tau+\frac{1}{\pi}\phi(t)\ln\frac{2-t}{1+t},\ t\in(-1,2).$$

Let $t_1,t_2\in[0,1]$, $0\le t_1<t_2\le 1$ $(0<t_2-t_1\le 1)$ and $\varepsilon=\frac{t_2-t_1}{2}$. In that case, we write

$$\pi\left(F\phi(t_2)-F\phi(t_1)\right)=\int_{-1}^{2}\frac{\phi(\tau)-\phi(t_2)}{\tau-t_2}d\tau-\int_{-1}^{2}\frac{\phi(\tau)-\phi(t_1)}{\tau-t_1}d\tau+\phi(t_2)\ln\frac{2-t_2}{1+t_2}-\phi(t_1)\ln\frac{2-t_1}{1+t_1}$$

$$=\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\phi(\tau)-\phi(t_2)}{\tau-t_2}d\tau-\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\phi(\tau)-\phi(t_1)}{\tau-t_1}d\tau+\int_{-1}^{t_1-\varepsilon}\left[\frac{\phi(\tau)-\phi(t_2)}{\tau-t_2}-\frac{\phi(\tau)-\phi(t_1)}{\tau-t_1}\right]d\tau$$

$$+\int_{t_2-\varepsilon}^{2}\left[\frac{\phi(\tau)-\phi(t_2)}{\tau-t_2}-\frac{\phi(\tau)-\phi(t_1)}{\tau-t_1}\right]d\tau+\phi(t_2)\ln\frac{2-t_2}{1+t_2}-\phi(t_1)\ln\frac{2-t_1}{1+t_1}$$

$$=\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\phi(\tau)-\phi(t_2)}{\tau-t_2}d\tau-\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\phi(\tau)-\phi(t_1)}{\tau-t_1}d\tau$$

$$+\int_{-1}^{t_1-\varepsilon}\left[\frac{\phi(\tau)-\phi(t_1)}{\tau-t_2}-\frac{\phi(\tau)-\phi(t_1)}{\tau-t_1}+\frac{\phi(t_1)-\phi(t_2)}{\tau-t_2}\right]d\tau$$

$$+\int_{t_2-\varepsilon}^{2}\left[\frac{\phi(\tau)-\phi(t_2)}{\tau-t_2}-\frac{\phi(\tau)-\phi(t_2)}{\tau-t_1}+\frac{\phi(t_1)-\phi(t_2)}{\tau-t_1}\right]d\tau$$

$$+\phi(t_2)\ln\frac{2-t_2}{1+t_2}-\phi(t_1)\ln\frac{2-t_1}{1+t_1}=\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\phi(\tau)-\phi(t_2)}{\tau-t_2}d\tau-\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\phi(\tau)-\phi(t_1)}{\tau-t_1}d\tau$$

$$+(t_2-t_1)\int_{-1}^{t_1-\varepsilon}\frac{\phi(\tau)-\phi(t_1)}{(\tau-t_1)(\tau-t_2)}d\tau+\left[\phi(t_1)-\phi(t_2)\right]\int_{-1}^{t_1-\varepsilon}\frac{d\tau}{\tau-t_2}$$

$$+(t_2-t_1)\int_{t_2-\varepsilon}^{2}\frac{\phi(\tau)-\phi(t_2)}{(\tau-t_1)(\tau-t_2)}d\tau+\left[\phi(t_1)-\phi(t_2)\right]\int_{t_2-\varepsilon}^{2}\frac{d\tau}{\tau-t_1}$$

$$+\phi(t_2)\ln\frac{2-t_2}{1+t_2}-\phi(t_1)\ln\frac{2-t_1}{1+t_1}=\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\phi(\tau)-\phi(t_2)}{\tau-t_2}d\tau-\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\phi(\tau)-\phi(t_1)}{\tau-t_1}d\tau$$

$$+(t_2-t_1)\int_{-1}^{t_1-\varepsilon}\frac{\phi(\tau)-\phi(t_1)}{(\tau-t_1)(\tau-t_2)}d\tau+(t_2-t_1)\int_{t_2-\varepsilon}^{2}\frac{\phi(\tau)-\phi(t_2)}{(\tau-t_1)(\tau-t_2)}d\tau$$

$$+\left[\phi(t_1)-\phi(t_2)\right]\left[\ln\left|t_1-\varepsilon-t_2\right|-\ln(1+t_2)\right]+\left[\phi(t_1)-\phi(t_2)\right]$$

$$\times\left[\ln(2-t_1)-\ln\left|t_2-\varepsilon-t_1\right|\right]+\phi(t_2)\ln\frac{2-t_2}{1+t_2}-\phi(t_1)\ln\frac{2-t_1}{1+t_1}.$$

As a result of simple calculations, we write

$$\left[\phi(t_1)-\phi(t_2)\right]\left[\ln\left|t_1-\varepsilon-t_2\right|-\ln(1+t_2)\right]+\left[\phi(t_1)-\phi(t_2)\right]\left[\ln(2-t_1)-\ln\left|t_2-\varepsilon-t_1\right|\right]$$

$$+\phi(t_2)\ln\frac{2-t_2}{1+t_2}-\phi(t_1)\ln\frac{2-t_1}{1+t_1}=\left[\phi(t_1)-\phi(t_2)\right]\left[\ln\frac{3}{2}(t_2-t_1)-\ln(1+t_2)\right]+\left[\phi(t_1)-\phi(t_2)\right]$$

$$\times\left[\ln(2-t_1)-\ln\frac{t_2-t_1}{2}\right]+\phi(t_2)\left[\ln(2-t_2)-\ln(1+t_2)\right]-\phi(t_1)\left[\ln(2-t_1)-\ln(1+t_1)\right]$$

$$=\left[\phi(t_1)-\phi(t_2)\right]\left[\ln 3+\ln\frac{t_2-t_1}{2}-\ln(1+t_2)\right]+\left[\phi(t_1)-\phi(t_2)\right]\left[\ln(2-t_1)-\ln\frac{t_2-t_1}{2}\right]$$

$$+\phi(t_2)\left[\ln(2-t_2)-\ln(1+t_2)\right]-\phi(t_1)\left[\ln(2-t_1)-\ln(1+t_1)\right]$$

$$=\phi(t_1)\ln\frac{1+t_1}{1+t_2}+\phi(t_2)\ln\frac{2-t_2}{2-t_1}+\left[\phi(t_1)-\phi(t_2)\right]\ln 3.$$

Thus, we obtain

$$\pi\left(F\phi(t_2)-F\phi(t_1)\right)=\sum_{\nu=1}^{7}I_\nu .$$

Here,

$$I_1=\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\phi(\tau)-\phi(t_2)}{\tau-t_2}d\tau,$$

$$I_2=-\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\phi(\tau)-\phi(t_1)}{\tau-t_1}d\tau,$$

$$I_3=(t_2-t_1)\int_{-1}^{t_1-\varepsilon}\frac{\phi(\tau)-\phi(t_1)}{(\tau-t_1)(\tau-t_2)}d\tau,$$

$$I_4=(t_2-t_1)\int_{t_2-\varepsilon}^{2}\frac{\phi(\tau)-\phi(t_2)}{(\tau-t_1)(\tau-t_2)}d\tau,$$

$$I_5=\phi(t_1)\ln\frac{1+t_1}{1+t_2},$$

$$I_6=\phi(t_2)\ln\frac{2-t_2}{2-t_1},$$

$$I_7=(\phi(t_1)-\phi(t_2))\ln 3.$$

Now we consider $|I_\nu|, \nu=1,\cdots,7.$

For I_1, we write

$$|I_1|\le\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\omega(\phi,|\tau-t_2|)}{|\tau-t_2|}d\tau=\int_{\varepsilon}^{\varepsilon+t_2-t_1}\frac{\omega(\phi,\xi)}{\xi}d\xi=\int_{0}^{t_2-t_1}\frac{\omega(\phi,\xi+\varepsilon)}{\xi+\varepsilon}d\xi\le\int_{0}^{t_2-t_1}\frac{\omega(\phi,\xi)}{\xi}d\xi.$$

Similarly,

$$|I_2|\le\int_{t_1-\varepsilon}^{t_2-\varepsilon}\frac{\omega(\phi,|\tau-t_1|)}{|\tau-t_1|}d\tau=\int_{t_1-\varepsilon}^{t_1}\frac{\omega(\phi,t_1-\tau)}{t_1-\tau}d\tau+\int_{t_1}^{t_2-\varepsilon}\frac{\omega(\phi,\tau-t_1)}{\tau-t_1}d\tau=2\int_{0}^{\varepsilon}\frac{\omega(\phi,\xi)}{\xi}d\xi$$
$$=2\int_{0}^{t_2-t_1}\frac{\omega(\phi,\xi/2)}{\xi}d\xi\le 2\int_{0}^{t_2-t_1}\frac{\omega(\phi,\xi)}{\xi}d\xi.$$

Also, we can write

$$|I_3|\le(t_2-t_1)\int_{-1}^{t_1-\varepsilon}\frac{\omega(\phi,t_1-\tau)}{(t_1-\tau)(t_2-\tau)}d\tau=(t_2-t_1)\int_{\varepsilon}^{1+t_1}\frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)}d\xi=\sum_{\nu=1}^{3}I_3^{(\nu)}.$$

Here,

$$I_3^{(1)}=(t_2-t_1)\int_{\varepsilon}^{t_2-t_1}\frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)}d\xi,$$

$$I_3^{(2)}=(t_2-t_1)\int_{t_2-t_1}^{1}\frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)}d\xi,$$

$$I_3^{(3)}=(t_2-t_1)\int_{1}^{1+t_1}\frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)}d\xi.$$

For $I_3^{(1)}$ and $I_3^{(2)}$, we write

$$I_3^{(1)} \le \int_{\varepsilon}^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi} d\xi = \int_0^{\varepsilon} \frac{\omega(\phi,\xi+\varepsilon)}{\xi+\varepsilon} d\xi \le \int_0^{\varepsilon} \frac{\omega(\phi,\xi)}{\xi} d\xi = \int_0^{t_2-t_1} \frac{\omega(\phi,\xi/2)}{\xi} d\xi \le \int_0^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi} d\xi,$$

$$I_3^{(2)} \le (t_2-t_1) \int_{t_2-t_1}^{1} \frac{\omega(\phi,\xi)}{\xi^2} d\xi.$$

For $I_3^{(3)}$, we have

$$I_3^{(3)} = (t_2-t_1) \int_1^{1+t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)} d\xi = (t_2-t_1) \int_0^{t_1} \frac{\omega(\phi,\xi+1)}{(\xi+1)(\xi+1+t_2-t_1)} d\xi$$
$$\le (t_2-t_1) \int_0^{t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)} d\xi.$$

Here, we need to consider two cases: $t_1 \le t_2 - t_1$ and $t_1 > t_2 - t_1$.

Case 1: If $t_1 \le t_2 - t_1$, then

$$I_3^{(3)} \le (t_2-t_1) \int_0^{t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)} d\xi \le (t_2-t_1) \int_0^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)} d\xi \le \int_0^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi} d\xi.$$

Case 2: If $t_1 > t_2 - t_1$, then

$$I_3^{(3)} \le (t_2-t_1) \int_0^{t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)} d\xi = (t_2-t_1) \int_0^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)} d\xi + (t_2-t_1) \int_{t_2-t_1}^{t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)} d\xi.$$

Furthermore, since we can write

$$(t_2-t_1) \int_0^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)} d\xi \le \int_0^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi} d\xi,$$

$$(t_2-t_1) \int_{t_2-t_1}^{t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)} d\xi \le (t_2-t_1) \int_{t_2-t_1}^{t_1} \frac{\omega(\phi,\xi)}{\xi^2} d\xi \le (t_2-t_1) \int_{t_2-t_1}^{1} \frac{\omega(\phi,\xi)}{\xi^2} d\xi,$$

we obtain

$$I_3^{(3)} \le \int_0^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi} d\xi + (t_2-t_1) \int_{t_2-t_1}^{1} \frac{\omega(\phi,\xi)}{\xi^2} d\xi.$$

Thus, considering estimates for $I_3^{(1)}, I_3^{(2)}$ and $I_3^{(3)}$, we obtain

$$|I_3| \le 2\left(\int_0^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi} d\xi + (t_2-t_1) \int_{t_2-t_1}^{1} \frac{\omega(\phi,\xi)}{\xi^2} d\xi\right).$$

Now we consider I_4. We can write

$$|I_4| \le (t_2-t_1) \int_{t_2-\varepsilon}^{2} \frac{\omega(\phi,|\tau-t_2|)}{|\tau-t_1||\tau-t_2|} d\tau = \sum_{\nu=1}^{3} J_4^{(\nu)}.$$

Here,

$$I_4^{(1)} = (t_2-t_1) \int_{t_2-\varepsilon}^{t_2} \frac{\omega(\phi,t_2-\tau)}{(t_2-\tau)(\tau-t_1)} d\tau,$$

$$I_4^{(2)} = (t_2-t_1) \int_{t_2}^{t_2+\varepsilon} \frac{\omega(\phi,\tau-t_2)}{(\tau-t_2)(\tau-t_1)} d\tau,$$

$$I_4^{(3)} = (t_2-t_1) \int_{t_2+\varepsilon}^{2} \frac{\omega(\phi,\tau-t_2)}{(\tau-t_2)(\tau-t_1)} d\tau.$$

For $I_4^{(1)}$, we obtain

$$I_4^{(1)} = (t_2 - t_1)\int_0^{\varepsilon} \frac{\omega(\phi,\xi)}{\xi(t_2 - t_1 - \xi)} d\xi = (t_2 - t_1)\int_{\varepsilon}^{t_2 - t_1} \frac{\omega(\phi, t_2 - t_1 - \xi)}{(t_2 - t_1 - \xi)\xi} d\xi$$
$$\leq 2\int_{\varepsilon}^{t_2 - t_1} \frac{\omega(\phi, t_2 - t_1 - \xi)}{t_2 - t_1 - \xi} d\xi = 2\int_0^{\varepsilon} \frac{\omega(\phi,\xi)}{\xi} d\xi = 2\int_0^{t_2 - t_1} \frac{\omega(\phi,\xi/2)}{\xi} d\xi \leq 2\int_0^{t_2 - t_1} \frac{\omega(\phi,\xi)}{\xi} d\xi.$$

Similarly,

$$I_4^{(2)} = (t_2 - t_1)\int_0^{\varepsilon} \frac{\omega(\phi,\xi)}{\xi(\xi + t_2 - t_1)} d\xi \leq \int_0^{\varepsilon} \frac{\omega(\phi,\xi)}{\xi} d\xi = \int_0^{t_2 - t_1} \frac{\omega(\phi,\xi/2)}{\xi} d\xi \leq \int_0^{t_2 - t_1} \frac{\omega(\phi,\xi)}{\xi} d\xi.$$

For $I_4^{(3)}$, we write

$$I_4^{(3)} = (t_2 - t_1)\int_{t_2 + \varepsilon}^{2} \frac{\omega(\phi, \tau - t_2)}{(\tau - t_2)(\tau - t_1)} d\tau = (t_2 - t_1)\int_{\varepsilon}^{2 - t_2} \frac{\omega(\phi,\xi)}{\xi(\xi + t_2 - t_1)} d\xi = \sum_{\nu=1}^{3} I_4^{(3,\nu)}.$$

Here,

$$I_4^{(3,1)} = (t_2 - t_1)\int_{\varepsilon}^{t_2 - t_1} \frac{\omega(\phi,\xi)}{\xi(\xi + t_2 - t_1)} d\xi,$$
$$I_4^{(3,2)} = (t_2 - t_1)\int_{t_2 - t_1}^{1} \frac{\omega(\phi,\xi)}{\xi(\xi + t_2 - t_1)} d\xi,$$
$$I_4^{(3,3)} = (t_2 - t_1)\int_{1}^{2 - t_2} \frac{\omega(\phi,\xi)}{\xi(\xi + t_2 - t_1)} d\xi.$$

For $I_4^{(3,1)}$ and $I_4^{(3,2)}$, we obtain following estimates

$$I_4^{(3,1)} = (t_2 - t_1)\int_{\varepsilon}^{t_2 - t_1} \frac{\omega(\phi,\xi)}{\xi(\xi + t_2 - t_1)} d\xi = (t_2 - t_1)\int_0^{\varepsilon} \frac{\omega(\phi,\xi + \varepsilon)}{(\xi + \varepsilon)(\xi + 3\varepsilon)} d\xi$$
$$\leq \frac{t_2 - t_1}{3\varepsilon}\int_0^{\varepsilon} \frac{\omega(\phi,\xi + \varepsilon)}{\xi + \varepsilon} d\xi = \frac{2}{3}\int_0^{\varepsilon} \frac{\omega(\phi,\xi + \varepsilon)}{\xi + \varepsilon} d\xi \leq \frac{2}{3}\int_0^{\varepsilon} \frac{\omega(\phi,\xi)}{\xi} d\xi$$
$$= \frac{2}{3}\int_0^{t_2 - t_1} \frac{\omega(\phi,\xi/2)}{\xi} d\xi \leq \frac{2}{3}\int_0^{t_2 - t_1} \frac{\omega(\phi,\xi)}{\xi} d\xi,$$
$$I_4^{(3,2)} = (t_2 - t_1)\int_{t_2 - t_1}^{1} \frac{\omega(\phi,\xi)}{\xi(\xi + t_2 - t_1)} d\xi \leq (t_2 - t_1)\int_{t_2 - t_1}^{1} \frac{\omega(\phi,\xi)}{\xi \cdot \xi} d\xi = (t_2 - t_1)\int_{t_2 - t_1}^{1} \frac{\omega(\phi,\xi)}{\xi^2} d\xi.$$

Now, we consider $I_4^{(3,3)}$. Firstly, we can write

$$I_4^{(3,3)} = (t_2 - t_1)\int_{1}^{2 - t_2} \frac{\omega(\phi,\xi)}{\xi(\xi + t_2 - t_1)} d\xi = (t_2 - t_1)\int_0^{1 - t_2} \frac{\omega(\phi,\xi + 1)}{(\xi + 1)(\xi + 1 + t_2 - t_1)} d\xi$$
$$\leq (t_2 - t_1)\int_0^{1 - t_2} \frac{\omega(\phi,\xi)}{\xi(\xi + t_2 - t_1)} d\xi.$$

Here, we need to consider two cases: $1 - t_2 \leq t_2 - t_1$ and $1 - t_2 > t_2 - t_1$.

Case 1: If $1-t_2 \le t_2 - t_1$, then

$$I_4^{(3,3)} \le (t_2-t_1)\int_0^{1-t_2} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)}d\xi \le (t_2-t_1)\int_0^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)}d\xi \le \int_0^{t_2-t_1}\frac{\omega(\phi,\xi)}{\xi}d\xi.$$

Case 2: If $1-t_2 > t_2 - t_1$, then

$$I_4^{(3,3)} \le (t_2-t_1)\int_0^{t_2-t_1} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)}d\xi + (t_2-t_1)\int_{t_2-t_1}^{1-t_2} \frac{\omega(\phi,\xi)}{\xi(\xi+t_2-t_1)}d\xi \le \int_0^{t_2-t_1}\frac{\omega(\phi,\xi)}{\xi}d\xi$$
$$+(t_2-t_1)\int_{t_2-t_1}^{1-t_2}\frac{\omega(\phi,\xi)}{\xi^2}d\xi \le \int_0^{t_2-t_1}\frac{\omega(\phi,\xi)}{\xi}d\xi + (t_2-t_1)\int_{t_2-t_1}^{1}\frac{\omega(\phi,\xi)}{\xi^2}d\xi.$$

Thus, using the results for $I_4^{(3,\nu)}, \nu = 1,2,3$, we obtain the following estimate:

$$I_4^{(3)} \le \frac{5}{3}\int_0^{t_2-t_1}\frac{\omega(\phi,\xi)}{\xi}d\xi + 2(t_2-t_1)\int_{t_2-t_1}^{1-t_2}\frac{\omega(\phi,\xi)}{\xi^2}d\xi.$$

Using the results for $I_4^{(\nu)}, \nu = 1,2,3$, we obtain the following estimate:

$$|I_4| \le \frac{14}{3}\int_0^{t_2-t_1}\frac{\omega(\phi,\xi)}{\xi}d\xi + 2(t_2-t_1)\int_{t_2-t_1}^{1}\frac{\omega(\phi,\xi)}{\xi^2}d\xi.$$

Now, we consider $I_\nu, \nu = 5,6,7$.

For I_5, we write

$$|I_5| = |\phi(t_1)|(\ln(1+t_2) - \ln(1+t_1)).$$

For the right-hand side, we use the mean value theorem on the interval $[1+t_1, 1+t_2]$ and obtain the following:

$$|I_5| = |\phi(t_1)|\frac{t_2-t_1}{2+t_1+\theta\cdot(t_2-t_1)} = |\phi(t_1)-\phi(-1)|\frac{t_2-t_1}{2+t_1+\theta\cdot(t_2-t_1)}, \theta\in[0,1].$$

Thus, we obtain the following estimate:

$$|I_5| \le \omega(\phi, 1+t_1)\frac{t_2-t_1}{2+t_1+\theta\cdot(t_2-t_1)} \le \frac{\omega(\phi, t_2-t_1)}{t_2-t_1}(1+t_1)\frac{t_2-t_1}{2+t_1+\theta\cdot(t_2-t_1)}$$
$$\le \frac{2}{2+t_1+\theta\cdot(t_2-t_1)}\omega(\phi,t_2-t_1) \le \omega(\phi, t_2-t_1).$$

Now, we consider the term $I_6 = \phi(t_2)\ln\frac{2-t_2}{2-t_1}$.

We rewrite the difference $\ln(2-t_1) - \ln(2-t_2)$ as follows:

$$\ln(2-t_1) - \ln(2-t_2) = \ln(1+(1-t_1)) - \ln(1+(1-t_2)).$$

We set $1-t_1 = w_1$, $1-t_2 = w_2$ and deduce that $w_1, w_2 \in [0,1]$ and $0 \le w_2 < w_1 \le 1$. Thus, we can write

$$\ln(2-t_1) - \ln(2-t_2) = \ln(1+w_1) - \ln(1+w_2).$$

Therefore,

$$|I_6| = |\phi(t_2)|(\ln(1+w_1) - \ln(1+w_2)).$$

Then, for the function $\ln(1+w)$, similarly to the previous term, we apply the mean value theorem on the interval $[1+w_2, 1+w_1]$ when $\theta\in[0,1]$,

$$|I_6| = |\phi(t_2)|\frac{w_1-w_2}{2+w_2+\theta\cdot(w_1-w_2)} = |\phi(t_2)-\phi(0)|\frac{w_1-w_2}{2+w_2+\theta\cdot(w_1-w_2)}.$$

Hence,

$$|I_6| = |\phi(t_2) - \phi(0)| \frac{t_2 - t_1}{2 + w_2 + \theta \cdot (t_2 - t_1)} \leq \frac{t_2 - t_1}{2} \omega(\phi, t_2)$$
$$\leq \frac{t_2 - t_1}{2} \frac{\omega(\phi, t_2 - t_1)}{t_2 - t_1} t_2 \leq \frac{\omega(\phi, t_2 - t_1)}{2}.$$

Also for I_7, we write

$$|I_7| = |\phi(t_1) - \phi(t_2)| \ln 3 \leq \omega(\phi, t_2 - t_1) \ln 3.$$

Hence, we obtain the following estimate:

$$|I_5| + |I_6| + |I_7| \leq (\frac{3}{2} + \ln 3) \omega(\phi, t_2 - t_1).$$

Furthermore, since we can write

$$\omega(\phi, t_2 - t_1) = \frac{\omega(\phi, t_2 - t_1)}{t_2 - t_1}(t_2 - t_1) = \int_0^{t_2 - t_1} \frac{\omega(\phi, t_2 - t_1)}{t_2 - t_1} d\xi,$$

we obtain

$$|I_5| + |I_6| + |I_7| \leq (\frac{3}{2} + \ln 3) \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi} d\xi.$$

However, since for every $x \in (0,1]$, $\omega(\varphi, x) = \omega(\phi, x)$ and for every $t \in [0,1]$, $F\phi(t) = S\varphi(t)$, using the previous observations, we can deduce that

$$|S\varphi(t_2) - S\varphi(t_1)| \leq c_1 Z(\omega(\varphi, \cdot), t_2 - t_1), c_1 = \frac{67}{6} + \ln 3. \quad (3.3)$$

Now, we consider the difference $S_+\varphi(t_2) - S_+\varphi(t_1)$, which we can write

$$S_+\varphi(t_2) - S_+\varphi(t_1) = \frac{1}{\pi} \int_0^{t_2 - t_1} \frac{\varphi(\tau)}{\tau + t_2} d\tau - \frac{1}{\pi} \int_0^{t_2 - t_1} \frac{\varphi(\tau)}{\tau + t_1} d\tau + \frac{t_1 - t_2}{\pi} \int_{t_2 - t_1}^{1} \frac{\varphi(\tau)}{(\tau + t_1)(\tau + t_2)} d\tau.$$

Since $|\varphi(\tau)| = |\varphi(\tau) - \varphi(0)| \leq \omega(\phi, \tau)$, we obtain

$$|S_+\varphi(t_2) - S_+\varphi(t_1)| \leq \frac{1}{\pi} \int_0^{t_2 - t_1} \frac{\omega(\varphi, \tau)}{\tau + t_2} d\tau + \frac{1}{\pi} \int_0^{t_2 - t_1} \frac{\omega(\varphi, \tau)}{\tau + t_1} d\tau + \frac{t_2 - t_1}{\pi} \int_{t_2 - t_1}^{1} \frac{\omega(\varphi, \tau)}{(\tau + t_1)(\tau + t_2)} d\tau$$
$$\leq \frac{2}{\pi} \int_0^{t_2 - t_1} \frac{\omega(\varphi, \xi)}{\xi} d\xi + \frac{t_2 - t_1}{\pi} \int_{t_2 - t_1}^{1} \frac{\omega(\varphi, \xi)}{\xi^2} d\xi \leq \frac{2}{\pi} Z(\omega(\varphi, \cdot), t_2 - t_1).$$

Thus

$$|S_+\varphi(t_2) - S_+\varphi(t_1)| \leq c_1 Z(\omega(\varphi, \cdot), t_2 - t_1), c_1 = \frac{2}{\pi}. \quad (3.4)$$

Since the function $Z(\omega(\varphi, \cdot), t)$, $t \in [0,1]$ is non-decreasing, from inequalities (3.3) and (3.4), we see that inequalities (3.1) and (3.2) exist.

Thus, the proof of Theorem 3.1 is complete.

Theorem 3.2. Let the operators $S : \overset{\circ}{H}_\alpha \to H_\alpha$ and $S_+ : \overset{\circ}{H}_\alpha \to H_\alpha$ be defined as in formula (1.2) and $\varphi \in \overset{\circ}{H}_\alpha$. Then,

$$\|S\|_\alpha \leq A(\alpha), \ \|S_+\|_\alpha \leq B(\alpha), \quad (3.5)$$

$$\|S\|_\infty \leq C(\alpha), \ \|S_+\|_\infty \leq D(\alpha). \quad (3.6)$$

Here, the constants $A(\alpha), B(\alpha), C(\alpha)$ and $D(\alpha)$ depend only on parameter α.

Proof. According to Theorem 3.1, for each $\varphi \in J_0$ and $x \in (0,1]$ we write

$$\omega(S\varphi, x) \leq c_1 Z(\omega(\varphi, \cdot), x) = c_1 (\int_0^x \frac{\omega(\varphi, \xi)}{\xi} d\xi + x \int_x^1 \frac{\omega(\varphi, \xi)}{\xi^2} d\xi)$$
$$= c_1 (\int_0^x \frac{\omega(\varphi, \xi)}{\xi^\alpha} \xi^{\alpha - 1} d\xi + x \int_x^1 \frac{\omega(\varphi, \xi)}{\xi^\alpha} \xi^{\alpha - 2} d\xi) \leq c_1 H(\varphi, \alpha) (\frac{x^\alpha}{\alpha} + x \cdot \frac{x^{\alpha - 1} - 1}{1 - \alpha})$$
$$= c_1 H(\varphi, \alpha) \frac{x^\alpha (1 - \alpha x^{1 - \alpha})}{\alpha(1 - \alpha)} \leq \frac{c_1}{\alpha(1 - \alpha)} x^\alpha H(\varphi, \alpha).$$

Thus,

$$\omega(S\varphi,x)\le\frac{c_1}{\alpha(1-\alpha)}x^\alpha H(\varphi,\alpha).$$

It follows that

$$H(S\varphi,\alpha)\le\frac{c_1}{\alpha(1-\alpha)}H(\varphi,\alpha)\le\frac{c_1}{\alpha(1-\alpha)}\|\varphi\|_{\alpha,0}. \quad (3.7)$$

Similarly

$$\omega(S_+\varphi,x)\le\frac{c_2}{\alpha(1-\alpha)}x^\alpha H(\varphi,\alpha)$$

and

$$H(S_+\varphi,\alpha)\le\frac{c_2}{\alpha(1-\alpha)}H(\varphi,\alpha)\le\frac{c_2}{\alpha(1-\alpha)}\|\varphi\|_{\alpha,0}. \quad (3.8)$$

From the definition of the operator $F\phi(t)$, we write

$$\pi|F\phi(t)|\le\int_{-1}^{2}\frac{\omega(\phi,|\tau-t|)}{|\tau-t|}d\tau+\omega(\phi,1+t)\ln 2=\int_{-1}^{t}\frac{\omega(\phi,t-\tau)}{t-\tau}d\tau+\int_{t}^{2}\frac{\omega(\phi,\tau-t)}{\tau-t}d\tau$$

$$+\omega(\phi,1+t)\ln 2=\int_0^{1+t}\frac{\omega(\phi,\xi)}{\xi}d\xi+\int_0^{2-t}\frac{\omega(\phi,\xi)}{\xi}d\xi+\omega(\phi,1+t)\ln 2$$

$$\le H(\varphi,\alpha)\Big(\int_0^{1+t}\xi^{\alpha-1}d\xi+\int_0^{2-t}\xi^{\alpha-1}d\xi+(1+t)^\alpha\ln 2\Big)$$

$$=H(\varphi,\alpha)\Big(\frac{(1+t)^\alpha+(2-t)^\alpha}{\alpha}+(1+t)^\alpha\ln 2\Big)\le 2^\alpha\Big(\frac{2}{\alpha}+\ln 2\Big)H(\varphi,\alpha).$$

According to this

$$\|S\varphi\|_\infty\le\frac{2^\alpha}{\pi}\Big(\frac{2}{\alpha}+\ln 2\Big)H(\varphi,\alpha)\le\frac{2^\alpha}{\pi}\Big(\frac{2}{\alpha}+\ln 2\Big)\|\varphi\|_{\alpha,0}. \quad (3.9)$$

Hence,

$$\|S\|_\infty\le C(\alpha), C(\alpha)=\frac{2^\alpha}{\pi}\Big(\frac{2}{\alpha}+\ln 2\Big).$$

From (3.7) and (3.9), we obtain

$$\|S\varphi\|_\alpha\le A(\alpha)\|\varphi\|_{\alpha,0}, A(\alpha)=\max\Big(\frac{c_1}{\alpha(1-\alpha)},C(\alpha)\Big).$$

Finally,

$$\|S\|_\alpha\le A(\alpha).$$

Now, we consider $\|S_+\varphi\|_\infty$.

Writing

$$S_+\varphi(t)=\frac{1}{\pi}\int_0^1\frac{\varphi(\tau)}{\tau+t}d\tau=\frac{1}{\pi}\int_0^1\frac{\varphi(\tau)-\varphi(0)}{\tau+t}d\tau=\frac{1}{\pi}\int_0^1\frac{\varphi(\tau)-\varphi(0)}{\tau^\alpha}\frac{\tau^\alpha}{\tau+t}d\tau$$

we obtain

$$|S_+\varphi(t)|\le\frac{H(\varphi,\alpha)}{\pi}\int_0^1\frac{\tau^\alpha}{\tau+t}d\tau\le\frac{H(\varphi,\alpha)}{\pi}\int_0^1\tau^{\alpha-1}d\tau\le\frac{1}{\alpha\pi}\|\varphi\|_{\alpha,0}.$$

Thus,

$$\|S_+\varphi\|_\infty\le\frac{1}{\alpha\pi}\|\varphi\|_{\alpha,0}. \quad (3.10)$$

Hence,

$$\|S_+\|_\infty\le D(\alpha), D(\alpha)=\frac{1}{\alpha\pi}.$$

From (3.8) and (3.10), we obtain

$$\|S_+\varphi\|_\alpha\le B(\alpha)\|\varphi\|_{\alpha,0}, B(\alpha)=\max\Big(\frac{c_2}{\alpha(1-\alpha)},D(\alpha)\Big).$$

Finally,

$$\|S_+\|_\alpha\le B(\alpha).$$

Thus, the proof of Theorem 3.2 is complete.

4. Discussion

Using Theorem 3.1 and Theorem 3.2, we can show that the operator

$$A\varphi(t)=f(t)\left\{\varphi^2(t)+\left[\lambda-S\varphi(t)+\mu\cdot S_+\varphi(t)\right]^2\right\},\ t\in[0,1]$$

is a contraction mapping. Furthermore, we can show that the operator A maps a closed sphere of space into itself. Thus, the conditions of the Banach contraction mapping principle are satisfied for the operator equation

$$\varphi(t) = A\varphi(t),\ t \in [0,1];$$

in other words, this equation has a unique solution.

5. Conclusion

In the investigation of the existence of solution of nonlinear singular integral equation

$$\varphi(t) = f(t)\left\{\varphi^2(t) + \left[\lambda - S\varphi(t) + \mu S_+\varphi(t)\right]^2\right\},\ t \in [0,1],$$

it is important to examine the singular integral operators, defined as follows:

$$S\varphi(t) = \frac{1}{\pi}\int_0^1 \frac{\varphi(\tau)}{\tau - t} d\tau \text{ and } S_+\varphi(t) = \frac{1}{\pi}\int_0^1 \frac{\varphi(\tau)}{\tau + t} d\tau. \quad (5.1)$$

Also, these integral operators have broad applications in the theory of elementary particles and scattering.

In this paper, we have examined some properties of singular integral operators, defined by (5.1). The main points of our conclusion are

1. To prove that the operators S and S_+ are bounded in the Holder space;
2. To give some important inequalities in the different norms for these operators.

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بعض المؤثرات التكاملية و خصائصها

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خلاصة

نقوم في هذا البحث براسة بعض المؤثرات التكاملية في فضاء هولدر، حيث لهذه المؤثرات تطبيقات واسعة في نظرية الجسميات الأساسية و نظرية التبعثر. ونثبت أن بعض المتباينات المهمة لمعيار بعض هذه المؤثرات تتحقق أيضاً في فضاء هولدر.