Some integral operators and their properties

Nizami Mustafa

Dept. of Mathematics, Faculty of Science and Letters, Kafkas University, 36100 Kars, Turkey
Email: nizamimustafa@gmail.com

Abstract

In this study, some integral operators, which have broad applications in the theory of elementary particles and scattering, have been investigated in Holder space. We show that some important inequalities for the norm of these operators are also satisfied in Holder space.

Keywords: Cauchy integral operator; integral operator.

AMS Subject Classifications: 47A63; 47B38; 47G10

1. Introduction

It is well known that the theory of singular integral equations has broad applications in many theoretical and practical investigations of problems in mathematics, mathematical physics, hydrodynamics and elasticity theory (Kalandia, 1973; Parton & Perlin, 1977; Colton & Kress, 1983; Duduchava, 1982; Panasyuk et al., 1984; Reşidoğlu, 2001).

Also, on the theoretical analysis of singular integral equations and on the approximate solution, there exist a lot of works (Ivanov, 1968; Lu, 1993; Mustafa, 2008; Mustafa & Khalilov, 2009; Mustafa & Caglar, 2010; Mustafa, 2013).

Consider the following nonlinear singular integral equation

\[ \varphi(t) = f(t) \left\{ \varphi^2(t) + \left[ \lambda - S\varphi(t) + \mu S_x\varphi(t) \right]^2 \right\}, \quad t \in [0,1], \] (1.1)

Here,

\[ S\varphi(t) = \frac{1}{\pi} \int_0^t \frac{\varphi(\tau)}{\tau - t} d\tau \quad \text{and} \quad S_x\varphi(t) = \frac{1}{\pi} \int_0^t \frac{\varphi(\tau)}{\tau + t} d\tau. \] (1.2)

Nonlinear singular integral equation (1.1) has crucial applications in the theory of elementary particles and scattering (Colton & Kress, 1983). It is important to examine the type of such equations. In the investigation of the existence of the solution of equation (1.1) it is important to examine the operators (1.2).

In this paper, we prove that the operators \( S \) and \( S_x \) from (1.2) are bounded in Holder space. Moreover, for these operators, some important inequalities in the different norms are also given.

2. Preliminaries

In this section, we will introduce some necessary information required for the proof of main results.

As usual, throughout the work, \( C[0,1] \) is the set of continuous functions defined on \( [0,1] \) with maximum norm

\[ \|f\|_\infty = \max \{ |f(t)| : t \in [0,1] \}. \]

Definition 2.1. (Daugavet, 1977). The function

\[ \omega(\varphi,x) = \sup \{ |\varphi(t_2) - \varphi(t_1)| : |t_2 - t_1| \leq x, x \in [0, +\infty) \} \]

is called the modulus of continuity of the bounded function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \).

Let us recall the properties of the modulus of continuity:

1. The modulus of continuity is a continuous function.
2. The modulus of continuity is a non-decreasing function.
3. For every \( x_1, x_2 > 0 \), \( \omega(x_1 + x_2) \leq \omega(x_1) + \omega(x_2) \).
4. \( \omega(0) = 0 \).
5. For every \( x > 0 \) and \( \alpha > 0 \) \( \frac{\omega(x)}{x^\alpha} \) and \( \frac{\omega(x)}{x^\alpha} \) are decreasing functions.

Definition 2.2. (Lu, 1993). Assume \( \varphi(t) \) is defined on \( [0,1] \). If
for arbitrary points \( x_i, x_j \in [0,1] \), where \( K > 0 \) and \( \alpha \) are definite constants, then \( \varphi(t) \) is said to satisfy the Holder condition of order \( \alpha \), or simply condition \( H_\alpha \), denoted by \( \varphi \in H_\alpha [0,1] \).

The functions in class \( H \) possess the following properties:

1. If \( \varphi \in H[0,1] \), then \( \varphi \in C[0,1] \), i.e., \( H[0,1] \subset C[0,1] \).
2. If \( \varphi \in H_\alpha [0,1] \) and \( 0 < \beta \leq \alpha \), then \( \varphi \in H_\beta [0,1] \), i.e., \( H_\alpha [0,1] \subset H_\beta [0,1] \), if \( 0 < \beta \leq \alpha \).
3. If both \( \varphi \) and \( \psi \in H_\alpha [0,1] \), then so do \( \varphi \pm \psi, \varphi \cdot \psi, \phi / \psi \) (\( \psi \neq 0 \) on \( [0,1] \)).
4. If both \( \varphi \) and \( \psi \) are in \( H_\alpha [0,1] \), then so do \( \varphi + \psi, \varphi \cdot \psi \) and

\[
H(\varphi + \psi; \alpha) = H(\varphi; \alpha) + H(\psi; \alpha)
\]

\[
H(\varphi \cdot \psi; \alpha) \leq H(\varphi; \alpha) \|\psi\|_{\alpha} + \|\phi\|_{\alpha} H(\psi; \alpha).
\]

The functions in class \( H \) possess the following properties:

1. If \( \varphi \in C[0,1] \), then \( \varphi \in C[0,1] \), i.e., \( H[0,1] \subset C[0,1] \).
2. If \( \varphi \in C[0,1] \) and \( 0 < \beta \leq \alpha \), then \( \varphi \in C_\beta [0,1] \), i.e., \( C_\alpha [0,1] \subset C_\beta [0,1] \), if \( 0 < \beta \leq \alpha \).
3. If both \( \varphi \) and \( \psi \in C_\alpha [0,1] \), then so do \( \varphi \pm \psi, \varphi \cdot \psi, \phi / \psi \) (\( \psi \neq 0 \) on \( [0,1] \)).
4. If both \( \varphi \) and \( \psi \) are in \( C_\alpha [0,1] \), then so do \( \varphi + \psi, \varphi \cdot \psi \) and

\[
H(\varphi + \psi; \alpha) = H(\varphi; \alpha) + H(\psi; \alpha)
\]

\[
H(\varphi \cdot \psi; \alpha) \leq H(\varphi; \alpha) \|\psi\|_{\alpha} + \|\phi\|_{\alpha} H(\psi; \alpha).
\]

Let \( C[0,1] = \{ f \in C[0,1] : f(0) = f(1) \} \) and \( C_\alpha [0,1] = \{ f \in C_\alpha [0,1] : \|f\|_{\alpha} = \phi(0) = \phi(1) \} \).

The function spaces \( H_\alpha [0,1] \) and \( C_\alpha [0,1] \) are Banach spaces with norm

\[
\|\varphi\|_{\alpha} = \max(\|\varphi\|_{\alpha}, H(\varphi; \alpha)).
\]

Here,

\[
H(\varphi; \alpha) = \sup \left\{ \frac{\omega(\varphi, x)}{x^\alpha} : 0 < x \leq 1 \right\}.
\]

Furthermore, throughout the paper, we denote \( H_\alpha (\hat{H}_\alpha) \) instead of \( H_\alpha [0,1] \), unless stated otherwise.

We denote the norm of function \( \varphi \in \hat{H}_\alpha \) by

\[
\|\varphi\|_{\alpha,0} = \max(\|\varphi\|_{\alpha}, H(\varphi; \alpha)).
\]

The norm of a bounded linear operator \( \mathcal{F} : \hat{H}_\alpha \to \hat{H}_\alpha \) is defined as follows (Kreyszig, 1978):

\[
\|\mathcal{F}\| = \sup_{\varphi \in \hat{H}_\alpha} \left\{ \mathcal{F} \|\varphi\|_{\alpha,0} : \varphi \in H_\alpha \right\}
\]

\[
\|\mathcal{F}\|_\alpha = \sup_{\varphi \in \hat{H}_\alpha \cap C[0,1]} \left\{ \mathcal{F} \|\varphi\|_{\alpha,0} : \varphi \in H_\alpha \right\}.
\]

Let

\[
J_\alpha = \left\{ \varphi \in C[0,1] : \int_0^1 \frac{\omega(\varphi, \xi)}{\xi} d\xi < +\infty \right\}
\]

and

\[
Z(\omega(\varphi, \cdot), t) = \int_0^t \frac{\omega(\varphi, \xi)}{\xi \xi^2} d\xi + \int_t^1 \frac{\omega(\varphi, \xi)}{\xi} d\xi, t \in [0,1].
\]

Then, the following is provided:

1. If \( \varphi \in J_\alpha \), then \( \varphi \in J_\alpha \), i.e., \( J_\alpha \subset J_\alpha \).
2. \( Z(\omega(\varphi, \cdot), t) \) is a non-decreasing function on \( [0,1] \).

3. Main results

In this section of the paper, we provide some properties of the operators \( S : \hat{H}_\alpha \to \hat{H}_\alpha \) and \( S_\alpha : \hat{H}_\alpha \to \hat{H}_\alpha \), which are defined in formula (1.2).

Theorem 3.1. Let the operators \( S : \hat{H}_\alpha \to \hat{H}_\alpha \) and \( S_\alpha : \hat{H}_\alpha \to \hat{H}_\alpha \) be defined as in formula (1.2) and \( \varphi \in J_\alpha \).

Then, for every \( x \in (0,1] \),

\[
\omega(S\varphi, x) \leq c_1 Z(\omega(\varphi, \cdot), x), \quad (3.1)
\]

\[
\omega(S_\varphi, x) \leq c_2 Z(\omega(\varphi, \cdot), x). \quad (3.2)
\]

Here, \( c_1 = \frac{1}{\pi} \left( \frac{67}{6} + \ln 3 \right) \), \( c_2 = \frac{2}{\pi} \) are definite constants.

Proof. Let

\[
\phi(t) = \begin{cases} \varphi(t) & \text{for } t \in [0,1], \\ 0 & \text{for } t \in [-1,2] \setminus [0,1] \end{cases}
\]

and

\[
F(\phi(t)) = \frac{1}{\pi} \int_{-t}^t \frac{\phi(t)}{\tau - t} d\tau, \quad t \in (-1,2).
\]

The operator \( F : \hat{H}_\alpha \to \hat{H}_\alpha \) can be written as

\[
F(\phi(t)) = \frac{1}{\pi} \int_{-t}^t \frac{\phi(t) - \phi(t)}{\tau - t} d\tau + \frac{1}{\pi} \phi(t) \ln \frac{2 - t}{1 + t}, \quad t \in (-1,2).
\]

Let \( t_1, t_2 \in [0,1], \quad 0 \leq t_1 < t_2 \leq 1 \) and \( 0 < t_2 - t_1 \leq 1 \) and \( e = \frac{t_2 - t_1}{2} \). In that case, we write
\[ \pi \left( F \phi(t_2) - F \phi(t_1) \right) = \frac{2}{t_2 - t_1} \int_{t_2}^{t_1} \phi(\tau) - \phi(t_1) \, d\tau - \frac{2}{t_2 - t_1} \int_{t_1}^{t_2} \phi(\tau) - \phi(t_2) \, d\tau + \phi(t_2) \ln \frac{2 - t_2}{1 + t_2} - \phi(t_1) \ln \frac{2 - t_1}{1 + t_1} \]

\[ = \int_{t_1}^{t_2} \phi(\tau) - \phi(t_1) \, d\tau - \int_{t_2}^{t_1} \phi(\tau) - \phi(t_2) \, d\tau + \int_{t_1}^{t_2} \left( \phi(\tau) - \phi(t_2) \right) - \left( \phi(\tau) - \phi(t_1) \right) \, d\tau \]

\[ + \int_{t_1}^{t_2} \left( \phi(\tau) - \phi(t_1) \right) \, d\tau + \int_{t_2}^{t_1} \left( \phi(\tau) - \phi(t_2) \right) \, d\tau \]

\[ + \phi(t_2) \ln \frac{2 - t_2}{1 + t_2} - \phi(t_1) \ln \frac{2 - t_1}{1 + t_1} \]

As a result of simple calculations, we write

\[ [\phi(t_1) - \phi(t_2)] \left[ \ln \left| t_1 - t_2 \right| - \ln(1 + t_2) \right] + [\phi(t_1) - \phi(t_2)] \left[ \ln(n - t_1) - \ln \left| t_2 - \varepsilon - t_1 \right| \right] \]

\[ + \phi(t_2) \ln \frac{2 - t_2}{1 + t_2} - \phi(t_1) \ln \frac{2 - t_1}{1 + t_1} = [\phi(t_1) - \phi(t_2)] \left[ \ln \left| t_1 - t_2 \right| - \ln(1 + t_2) \right] + [\phi(t_1) - \phi(t_2)] \]

\[ \times \left[ \ln(2 - t_1) - \ln \frac{t_1 - t_2}{2} \right] + \phi(t_2) \left[ \ln(2 - t_2) - \ln(1 + t_2) \right] - \phi(t_1) \left[ \ln(2 - t_1) - \ln(1 + t_1) \right] \]

\[ = [\phi(t_1) - \phi(t_2)] \left[ \ln 3 + \ln \frac{t_2 - t_1}{2} - \ln(1 + t_2) \right] + [\phi(t_1) - \phi(t_2)] \left[ \ln(2 - t_1) - \ln \frac{t_2 - t_1}{2} \right] \]

\[ + \phi(t_2) \left[ \ln(2 - t_2) - \ln(1 + t_2) \right] - \phi(t_1) \left[ \ln(2 - t_1) - \ln(1 + t_1) \right] \]

\[ = \phi(t_1) \ln \frac{1 + t_1}{1 + t_2} + \phi(t_2) \ln \frac{2 - t_2}{2 - t_1} + [\phi(t_1) - \phi(t_2)] \ln 3. \]
Thus, we obtain

\[ \pi \left( F\phi(t_2) - F\phi(t_1) \right) = \sum_{\nu=1}^{7} I_{\nu}. \]

Here,

\[ I_1 = \int_{t_2 - \varepsilon}^{t_2 + \varepsilon} \frac{\phi(\tau) - \phi(t_2)}{\tau - t_2} d\tau, \]

\[ I_2 = \int_{t_2 - \varepsilon}^{t_2 + \varepsilon} \frac{\phi(\tau) - \phi(t_1)}{\tau - t_1} d\tau, \]

\[ I_3 = (t_2 - t_1) \int_{1/2}^{1/3} \frac{\phi(\tau) - \phi(t_1)}{(\tau - t_1)(\tau - t_2)} d\tau, \]

\[ I_4 = (t_2 - t_1) \int_{1/2}^{1/3} \frac{\phi(\tau) - \phi(t_2)}{(\tau - t_1)(\tau - t_2)} d\tau, \]

\[ I_5 = \phi(t_1) \ln \frac{1 + t_1}{1 + t_2}, \]

\[ I_6 = \phi(t_2) \ln \frac{2 - t_2}{2 - t_1}, \]

\[ I_7 = (\phi(t_1) - \phi(t_2)) \ln 3. \]

Now we consider \(|I_\nu|, \nu = 1, \ldots, 7\).

For \(I_1\), we write

\[ |I_1| \leq \int_{\varepsilon}^{\varepsilon + \gamma} \frac{\omega(\phi, \tau - t_2)}{\tau - t_2} d\tau = \int_{\varepsilon}^{\varepsilon + \gamma} \frac{\omega(\phi, \xi)}{\xi} d\xi = \int_{0}^{\gamma} \frac{\omega(\phi, \xi + \varepsilon)}{\xi + \varepsilon} d\xi \leq \int_{0}^{\gamma} \frac{\omega(\phi, \xi)}{\xi} d\xi. \]

Similarly,

\[ |I_2| \leq \int_{\varepsilon}^{\varepsilon + \gamma} \frac{\omega(\phi, \tau - t_1)}{\tau - t_1} d\tau = \int_{\varepsilon}^{\varepsilon + \gamma} \frac{\omega(\phi, t_1 - \tau)}{t_1 - \tau} d\tau + \int_{t_1 - \varepsilon}^{t_1 - \varepsilon} \frac{\omega(\phi, \tau - t_1)}{\tau - t_1} d\tau = \int_{0}^{\gamma} \frac{\omega(\phi, \xi)}{\xi} d\xi. \]

Also, we can write

\[ |I_3| \leq (t_2 - t_1) \int_{1/2}^{1/3} \frac{\omega(\phi, t_1 - \tau)}{(t_1 - \tau)(t_2 - \tau)} d\tau = (t_2 - t_1) \int_{1/2}^{1/3} \frac{\omega(\phi, \xi)}{(\xi + t_2 - t_1)} d\xi = \sum_{\nu=1}^{3} I_{\nu}^{(v)}. \]

Here,

\[ I_3^{(1)} = (t_2 - t_1) \int_{\varepsilon}^{1/2} \frac{\omega(\phi, \xi)}{(\xi + t_2 - t_1)} d\xi, \]

\[ I_3^{(2)} = (t_2 - t_1) \int_{t_2 - t_1}^{1} \frac{\omega(\phi, \xi)}{(\xi + t_2 - t_1)} d\xi, \]

\[ I_3^{(3)} = (t_2 - t_1) \int_{1}^{1 - t_1/2} \frac{\omega(\phi, \xi)}{(\xi + t_2 - t_1)} d\xi. \]
For \( I_1^{(1)} \) and \( I_1^{(2)} \), we write

\[
I_1^{(1)} \leq \int_{\epsilon}^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi = \int_{0}^{\gamma_{1,h}} \frac{\omega(\phi, \xi + \epsilon)}{\xi + \epsilon} d\xi \leq \int_{0}^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi \leq \int_{0}^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi,
\]

\[
I_1^{(2)} \leq (t_2 - t_1) \int_{t_2 - \gamma}^{1} \frac{\omega(\phi, \xi)}{\xi^2} d\xi.
\]

For \( I_1^{(3)} \), we have

\[
I_1^{(3)} = (t_2 - t_1) \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi = (t_2 - t_1) \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi + 1)}{(\xi + 1)(\xi + 1 + t_2 - t_1)} d\xi
\]

\[
\leq (t_2 - t_1) \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi.
\]

Here, we need to consider two cases: \( t_1 \leq t_2 - t_1 \) and \( t_1 > t_2 - t_1 \).

Case 1: If \( t_1 \leq t_2 - t_1 \), then

\[
I_1^{(3)} \leq (t_2 - t_1) \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi \leq (t_2 - t_1) \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi + t_2 - t_1} d\xi \leq \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi.
\]

Case 2: If \( t_1 > t_2 - t_1 \), then

\[
I_1^{(3)} \leq (t_2 - t_1) \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi = (t_2 - t_1) \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi + t_2 - t_1} d\xi + (t_2 - t_1) \int_{t_1 - \gamma}^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi.
\]

Furthermore, since we can write

\[
(t_2 - t_1) \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi \leq (t_2 - t_1) \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi + t_2 - t_1} d\xi,
\]

\[
(t_2 - t_1) \int_{t_1 - \gamma}^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi \leq (t_2 - t_1) \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi + t_2 - t_1} d\xi,
\]

we obtain

\[
I_1^{(3)} \leq \int_0^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi} d\xi + (t_2 - t_1) \int_{t_1 - \gamma}^{\gamma_{1,h}} \frac{\omega(\phi, \xi)}{\xi^2} d\xi.
\]

Thus, considering estimates for \( I_1^{(3)} \), \( I_1^{(3)} \) and \( I_1^{(5)} \), we obtain

\[
|I_4| \leq (t_2 - t_1) \int_{t_1 - \gamma}^{\gamma_{1,h}} \frac{\omega(\phi, t_2 - t)}{|t - t_1|} d\tau = \sum_{v=1}^{3} I_4^{(v)}.
\]

Here,

\[
I_4^{(1)} = (t_2 - t_1) \int_{t_1 - \gamma}^{\gamma_{1,h}} \frac{\omega(\phi, t_2 - t)}{(t_2 - t_1) - t} d\tau,
\]

\[
I_4^{(2)} = (t_2 - t_1) \int_{t_2}^{\gamma_{1,h}} \frac{\omega(\phi, t_2 - t)}{(t_2 - t_1) - (t_2 - t)} d\tau,
\]

\[
I_4^{(3)} = (t_2 - t_1) \int_{t_1 - \gamma}^{\gamma_{1,h}} \frac{\omega(\phi, t_2 - t)}{(t_2 - t_1) - (t_2 - t)} d\tau.
\]

Now we consider \( I_4 \). We can write
For $I_4^{(1)}$, we obtain
\[ I_4^{(1)} = (t_2 - t_1)^6 \int_{0}^{6} \frac{\omega(\phi, \xi)}{\xi(t_2 - t_1 - \xi)} d\xi = (t_2 - t_1)^6 \int_{0}^{6} \frac{\omega(\phi, t_2 - t_1 - \xi)}{(t_2 - t_1 - \xi)^2} d\xi \]
\[ \leq 2 \int_{0}^{6} \frac{\omega(\phi, t_2 - t_1 - \xi)}{t_2 - t_1 - \xi} d\xi = 2 \int_{0}^{6} \frac{\omega(\phi, \xi)}{\xi} d\xi \leq 2 \int_{0}^{6} \frac{\omega(\phi, \xi/2)}{\xi/2} d\xi \]

Similarly,
\[ I_4^{(2)} = (t_2 - t_1)^6 \int_{0}^{6} \frac{\omega(\phi, \xi)}{\xi(\xi + t_2 - t_1)} d\xi \leq \int_{0}^{6} \frac{\omega(\phi, \xi)}{\xi} d\xi \leq \int_{0}^{6} \frac{\omega(\phi, \xi/2)}{\xi/2} d\xi \]

For $I_4^{(3)}$, we write
\[ I_4^{(3)} = (t_2 - t_1)^2 \int_{t_2}^{t_2 + \varepsilon} \frac{\omega(\phi, \tau - t_2)}{\tau - t_2}(\tau - t_1) \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi)}{\xi} d\xi \]
Here,
\[ I_4^{(3,1)} = (t_2 - t_1)^2 \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi)}{\xi} d\xi, \]
\[ I_4^{(3,2)} = (t_2 - t_1)^2 \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi/2)}{\xi/2} d\xi, \]

For $I_4^{(3,1)}$ and $I_4^{(3,2)}$, we obtain following estimates
\[ I_4^{(3,1)} = (t_2 - t_1)^2 \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi + \varepsilon)}{\xi + \varepsilon} d\xi \]
\[ \leq \frac{t_2 - t_1}{3\varepsilon} \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi)}{\xi} d\xi \leq \frac{2\varepsilon}{3} \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi)}{\xi} d\xi \]
\[ I_4^{(3,2)} = (t_2 - t_1)^2 \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi/2)}{\xi/2} d\xi \]
\[ \leq \frac{t_2 - t_1}{3\varepsilon} \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi/2)}{\xi/2} d\xi \leq \frac{2\varepsilon}{3} \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi)}{\xi} d\xi \]

Now, we consider $I_4^{(3,3)}$. Firstly, we can write
\[ I_4^{(3,3)} = (t_2 - t_1)^2 \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi)}{\xi} d\xi = (t_2 - t_1)^2 \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi + 1)}{(\xi + 1)(\xi + 1 + t_2 - t_1)} d\xi \]
\[ \leq (t_2 - t_1)^2 \int_{0}^{2\varepsilon} \frac{\omega(\phi, \xi)}{\xi} d\xi \]
Here, we need to consider two cases: $1 - t_2 \leq t_2 - t_1$ and $1 - t_2 > t_2 - t_1$. 

Case 1: If $1 - t_2 \leq t_2 - t_1$, then

$$I_4^{(3,3)} \leq (t_2 - t_1) \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi + t_2 - t_1} d\xi \leq (t_2 - t_1) \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi + t_2 - t_1} d\xi \leq \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi} d\xi.$$ 

Case 2: If $1 - t_2 > t_2 - t_1$, then

$$I_4^{(3,3)} \leq (t_2 - t_1) \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi + t_2 - t_1} d\xi + (t_2 - t_1) \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi + t_2 - t_1} d\xi \leq \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi} d\xi + (t_2 - t_1) \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi} d\xi.$$

Thus, using the results for $I_4^{(3,3)}, \nu = 1, 2, 3$, we obtain the following estimate:

$$I_4^{(3)} \leq \frac{\xi}{3} \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi} d\xi + 2(t_2 - t_1) \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi^2} d\xi.$$

Using the results for $I_4^{(\nu)}, \nu = 1, 2, 3$, we obtain the following estimate:

$$|I_4| \leq \frac{14}{3} t_2 - t_1 \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi} d\xi + 2(t_2 - t_1) \int_0^{t_2 - t_1} \frac{\omega(\phi, \xi)}{\xi^2} d\xi.$$

Now, we consider $I_5, I_6 = 5, 6, 7$.

For $I_5$, we write

$$|I_5| = |\phi(t_2)| \ln(1 + t_2) - \ln(1 + t_1).$$

For the right-hand side, we use the mean value theorem on the interval $[1 + t_1, 1 + t_2]$ and obtain the following:

$$|I_5| = |\phi(t_2)| \frac{t_2 - t_1}{2 + t_1 + \theta \cdot (t_2 - t_1)} = |\phi(t_2) - \phi(-1)| \frac{t_2 - t_1}{2 + t_1 + \theta \cdot (t_2 - t_1)}, \theta \in [0, 1].$$

Thus, we obtain the following estimate:

$$|I_5| \leq \omega(\phi, 1 + t_1) \frac{t_2 - t_1}{2 + t_1 + \theta \cdot (t_2 - t_1)} \leq \omega(\phi, t_2 - t_1) \frac{t_2 - t_1}{2 + t_1 + \theta \cdot (t_2 - t_1)} \leq \frac{2}{2 + t_1 + \theta \cdot (t_2 - t_1)} \omega(\phi, t_2 - t_1) \leq \omega(\phi, t_2 - t_1).$$

Now, we consider the term $I_6 = \phi(t_2) \ln \frac{2 - t_2}{2 - t_1}$. We rewrite the difference $\ln(2 - t_1) - \ln(2 - t_2)$ as follows:

$$\ln(2 - t_1) - \ln(2 - t_2) = \ln(1 + (1 - t_1)) - \ln(1 + (1 - t_2)).$$

We set $1 - t_1 = w_1, 1 - t_2 = w_2$ and deduce that $w_1, w_2 \in [0, 1]$ and $0 \leq w_2 < w_1 \leq 1$. Thus, we can write

$$|I_6| = \frac{w_1 - w_2}{2 + w_2 + \theta \cdot (w_1 - w_2)} = \frac{w_1 - w_2}{2 + w_2 + \theta \cdot (w_1 - w_2)}.$$
Hence,

\[ |I_1| = \left| \phi(t_2) - \phi(0) \right| \frac{t_2 - t_1}{2 + \theta(t_2 - t_1)} \leq \frac{t_2 - t_1}{2} \omega(\phi, t_2) \]

\[ \leq \frac{t_2 - t_1}{2} \omega(\phi, t_2 - t_1) t_2 \leq \omega(\phi, t_2 - t_1) \frac{t_2 - t_1}{2} . \]

Also for \( I_2 \), we write

\[ |I_2| = |\phi(t_2) - \phi(t_1)| \ln 3 \leq \omega(\phi, t_2 - t_1) \ln 3. \]

Hence, we obtain the following estimate:

\[ |I_1| + |I_2| = \left( \frac{3}{2} \right) \omega(\phi, t_2 - t_1) . \]

Furthermore, since we can write

\[ S \omega(t_2) - S \omega(t_1) = \frac{1}{\pi} \int_0^{t_2} \frac{\varphi(\tau) d\tau}{\tau + t_2} - \frac{1}{\pi} \int_0^{t_2} \frac{\varphi(\tau) d\tau}{\tau + t_1} + \frac{t_2 - t_1}{\pi} \frac{1}{(\tau + t_1)(\tau + t_2)} \int_0^{\tau_1} \varphi(\tau) d\tau . \]

Since \( |\varphi(\tau)| = |\varphi(\tau) - \varphi(0)| \leq \omega(\phi, \tau) \), we obtain

\[ |S \omega(t_2) - S \omega(t_1)| \leq \frac{1}{\pi} \int_0^{t_2} \frac{\omega(\phi, \tau) d\tau}{\tau + t_2} + \frac{1}{\pi} \int_0^{t_2} \frac{\omega(\phi, \tau) d\tau}{\tau + t_1} + \frac{t_2 - t_1}{\pi} \frac{1}{(\tau + t_1)(\tau + t_2)} \int_0^{\tau_1} \omega(\phi, \tau) d\tau . \]

Thus

\[ |S \omega(t_2) - S \omega(t_1)| \leq c_1 Z(\omega(\phi), t_2 - t_1), c_1 = \frac{2}{\pi}. \]  

Since the function \( Z(\omega(\phi), t) \), \( t \in [0, 1] \) is non-decreasing, from inequalities (3.3) and (3.4), we see that inequalities (3.1) and (3.2) exist.

Thus, the proof of Theorem 3.1 is complete.

Theorem 3.2. Let the operators \( S : H_a \rightarrow H_a \) and \( S_\omega : H_a \rightarrow H_a \) be defined as in formula (1.2) and \( \varphi \in H_a \).

Proof. According to Theorem 3.1, for each \( \varphi \in J_\alpha \) and \( x \in (0, 1] \) we write

\[ \omega(S \varphi, x) \leq c_1 Z(\omega(\phi), x) = c_1 \left( \int_0^x \frac{\omega(\phi, \xi)}{\xi} d\xi + \int_x^1 \frac{\omega(\phi, \xi)}{\xi^2} d\xi \right) \]

\[ = c_1 \left( \int_0^x \frac{\omega(\phi, \xi)}{\xi^\alpha} d\xi + \int_x^1 \frac{\omega(\phi, \xi)}{\xi^{\alpha - 1}} d\xi \right) \leq c_1 H(\varphi, \alpha) \frac{x^\alpha (1 - \alpha x^{\alpha - 1})}{\alpha (1 - \alpha)}. \]
Thus, \[
\omega(S\varphi, x) \leq \frac{c_1}{\alpha(1-\alpha)} x^\alpha H(\varphi, \alpha).
\]

It follows that
\[
H(S\varphi, \alpha) \leq \frac{c_1}{\alpha(1-\alpha)} H(\varphi, \alpha) \leq \frac{c_1}{\alpha(1-\alpha)} \|\varphi\|_{\alpha,0}. \quad (3.7)
\]

Similarly and
\[
H(S\varphi, \alpha) \leq \frac{c_2}{\alpha(1-\alpha)} H(\varphi, \alpha) \leq \frac{c_2}{\alpha(1-\alpha)} \|\varphi\|_{\alpha,0}. \quad (3.8)
\]

From the definition of the operator \(F\phi(t)\), we write
\[
\pi |F\phi(t)| \leq \int_{-1}^{1} \frac{\omega(\phi, |t-t|)}{|t-t|} d\tau + \omega(\phi, 1+t) \ln 2 = \int_{-1}^{1} \frac{\omega(\phi, t-t)}{t-t} d\tau + \int_{-1}^{1} \frac{\omega(\phi, |t-t|)}{t-t} d\tau + \omega(\phi, 1+t) \ln 2
\]
\[
\leq H(\varphi, \alpha) \left( \int_{0}^{1} \xi^{\alpha-1} d\xi + \int_{0}^{1} \xi^{\alpha-1} d\xi + (1+t)^\alpha \ln 2 \right)
\]
\[
= H(\varphi, \alpha) \left( \frac{(1+t)^\alpha + (2-t)^\alpha}{\alpha} + (1+t)^\alpha \ln 2 \right) \leq 2^\alpha \left( \frac{2}{\alpha} + \ln 2 \right) H(\varphi, \alpha).
\]

According to this
\[
\|S\varphi\| \leq \frac{2^\alpha}{\pi} \left( \frac{2}{\alpha} + \ln 2 \right) H(\varphi, \alpha) \leq \frac{2^\alpha}{\pi} \frac{2}{\alpha} \left( \frac{2}{\alpha} + \ln 2 \right) \|\varphi\|_{\alpha,0}. \quad (3.9)
\]

Hence,
\[
\|S\| \leq C(\alpha), C(\alpha) = \frac{2^\alpha}{\pi} \frac{2}{\alpha} (\frac{2}{\alpha} + \ln 2).
\]

From (3.7) and (3.9), we obtain

Writing
\[
S\varphi(t) = \frac{1}{\pi} \int_{0}^{1} \varphi(t) d\tau = \frac{1}{\pi} \int_{0}^{1} \frac{\varphi(t) - \varphi(0)}{t+t} d\tau = \frac{1}{\pi} \int_{0}^{1} \frac{\varphi(t) - \varphi(0)}{t+t} \frac{\tau^\alpha}{\tau} d\tau
\]

we obtain
\[
\|S\varphi\| \leq \frac{H(\varphi, \alpha)}{\pi} \int_{0}^{1} \frac{\tau^\alpha}{\tau+t} d\tau \leq \frac{H(\varphi, \alpha)}{\pi} \int_{0}^{1} \tau^{\alpha-1} d\tau \leq \frac{1}{\alpha \pi} \|\varphi\|_{\alpha,0}.
\]

Thus,
\[
\|S\varphi\| \leq \frac{1}{\alpha \pi} \|\varphi\|_{\alpha,0}. \quad (3.10)
\]

Finally,
\[
\|S\| \leq B(\alpha).
\]

Thus, the proof of Theorem 3.2 is complete.

4. Discussion

Using Theorem 3.1 and Theorem 3.2, we can show that the operator
\[
A\varphi(t) = f(t) \left\{ \varphi^2(t) + \left[ \lambda + S\varphi(t) + \mu \cdot S\varphi(t) \right] \right\}, \quad t \in [0,1]
\]
is a contraction mapping. Furthermore, we can show that the operator \(A\) maps a closed sphere of space into itself. Thus, the conditions of the Banach contraction mapping principle are satisfied for the operator equation

\[
\varphi(t) = A\varphi(t), \quad t \in [0,1];
\]

in other words, this equation has a unique solution.

5. Conclusion

In the investigation of the existence of solution of nonlinear singular integral equation

\[
\varphi(t) = f(t)\left\{q^{2}(t) + \left[\lambda - S\varphi(t) + \mu S_{\varphi}(t)\right]^2\right\}, \quad t \in [0,1],
\]

it is important to examine the singular integral operators, defined as follows:

\[
S\varphi(t) = \frac{1}{\pi} \int_{0}^{1} \frac{\varphi(\tau)}{\tau - t} \, d\tau \quad \text{and} \quad S_{\varphi}(t) = \frac{1}{\pi} \int_{0}^{1} \frac{\varphi(\tau)}{\tau + t} \, d\tau.
\]  

(5.1)

Also, these integral operators have broad applications in the theory of elementary particles and scattering.

In this paper, we have examined some properties of singular integral operators, defined by (5.1). The main points of our conclusion are

1. To prove that the operators \(S\) and \(S_{\varphi}\) are bounded in the Holder space;
2. To give some important inequalities in the different norms for these operators.

References


Submitted : 05/03/2015
Revised : 19/08/2015
Accepted : 20/01/2016
بعض المؤثرات التكاملية وخصائصها

نظامي مصطفى
قسم الرياضيات، كلية العلوم والأدب، جامعة الكفاكس، 36100 كارس، تركيا
المؤلف: nizamimustafa@gmail.com

خلاصة

تقوم في هذا البحث بدراسة بعض المؤثرات التكاملية في فضاء هولدر، حيث لهذه المؤثرات تطبيقات واسعة في نظرية الجسيمات الأساسية ونظرية التبخر. وثبت أن بعض المتباقات المهمة لمعيار بعض هذه المؤثرات تتحقق أيضاً في فضاء هولدر.