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A new type of convergence for a sequence of rays

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Abstract

In this work, we introduce the concept of rough convergence for a sequence of rays and obtain some basic results. In this context, if we take $r = 0$ then we obtain the classical results in the theory of rays.

Keywords: Rough convergence; rough limit set; sequence of rays.

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1. Introduction

In classical mathematical analysis, it is important to consider the convergence properties of a bounded sequence $\{x^\nu\} \subset \mathrm{R}^n$. If the sequence $\{x^\nu\}$ is unbounded, we are led to consider the concept of directions. That is why we deal with rays. Fenchel (1953) introduced the concept of the convergence of a sequence of rays. Fenchel (1953) also defined a metric by using the distance between the two rays. Moreover, he gave the definition of asymptotic cone, which is very important in optimization theory. Wijsman (1966) stated an equivalent definition of the asymptotic cone by means of the normalized sequences. Wijsman (1966) also proved that, if a sequence of convex sets is convergent to a set and the origin is contained in the closure of the limit set, then the sequence of projecting cones of these sets is convergent to the projecting cone of the limit set. In order to prove this theorem, he needed to introduce the notion of the asymptotic cone. Therefore, the asymptotic cones play an important role in the theory of convex sets.

The notion of rough convergence of a sequence was first given by Phu (2001) in finite dimensional normed spaces. Phu (2001) showed that a sequence, which is not convergent in the usual sense can be convergent to a point with a certain degree of roughness. He also proved that the rough limit set is convex, closed and bounded, and its diameter is smaller than $2r$. Subsequently, Phu (2003) has proved analogous results for infinite-dimensional spaces. In 2008, Aytar (2008) investigated the relations between the core and the r-limit set of a real sequence. Listán-García & Rambla-Barreno (2011) gave some results analogous to those of Phu, which are given by using strict convexity and uniform convexity, by means of uniform rotundity in every direction (URED). The condition URED is strictly weaker than the uniform convexity property. Listán-García & Rambla-Barreno (2014) stated two new geometric properties by using the rough convergence in Banach spaces. They also studied with Chebyshev centers and the M property of Kalton. Sudip *et al*. (2013) gave an extension of rough convergence, by using the notion of an ideal. They also stated some basic results related to the rough ideal limit set. Dündar & Çakan (2014) gave the definition of the rough convergence for a double sequence. In the recent literature, we see different types of convergence for sequences, such as convergence of order α (Et *et al*., 2014a; Et *et al*., 2014b).

In this paper, we apply the theory of rough convergence to the sequence of rays. We show that the sequence of rays is convergent to the whole R^n for $r \geq 2$. We state that some results in the classical analysis also hold for the sequence of rays. We also present the additive properties of the sequences of rays.

2. Preliminaries

Let R^n be an $n-$dimensional Euclidean vector space with origin 0, vectors (elements) $x, y, \ldots$, inner product

$\langle x, y\rangle$, norm $\|x\|=\sqrt{\langle x,x\rangle}$ and metric $d(x,y)=\|x-y\|$. Identify the vector x with the $n-$tuple $(x_1,x_2,\ldots,x_n)$ in R^n. Then $\langle x, y\rangle=\sum_{i=1}^{n} x_i y_i$ (Fenchel, 1953).

A subset M of R^n is called a *cone* if 0 is in M and $x\in M$ implies $\lambda x\in M$ for every non-negative real scalar λ. The particular cones consisting of a non-zero vector x and all its multiples λx $(\lambda\geq 0)$ are *rays*. A cone which contains at least one non-zero vector is therefore just the union of the rays it contains (Fenchel, 1953).

We state that $M=\{0\}$ is a cone in R^n. However, we will study with cones except for such cones.

Throughout the paper, (R^n) will denote the set of all rays in R^n, and a sequence of rays in (R^n) will be denoted by $\{(x^\nu)\}$.

Since cones may be thought of as sets of rays, it is desirable to introduce a topology on these rays from the topology on R^n. This might be done by defining the angle

$$\theta(x,y)=\arccos\frac{\langle x,y\rangle}{\|x\|\|y\|}, 0\leq\theta\leq\pi$$

as a metric on $\mathrm{R}^n-\{0\}$. This angle depends only on the rays (x) and (y) to which x and y belong. It may be thought of as the angle between the two rays. The proof that this angle is indeed a metric for the rays, in particular that it satisfies the triangle inequality is not obvious. An equivalent metric is given by

$$[x,y]=\sqrt{2-\frac{2\langle x,y\rangle}{\|x\|\|y\|}}.$$

This new metric is the *chord distance* between the two points $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$ on the unit sphere. That is, $[x,y]=d\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right)$. Clearly $[x,y]$ depends only on the rays (x) and (y). $[x,y]$ also satisfies the defining conditions for a metric on the space of rays. The geometric description shows that the two metrics are topologically equivalent (Fenchel, 1953).

A sequence $\{(x^\nu)\}$ is said to be convergent to a ray (x) if $[x^\nu,x]\to 0$ as $\nu\to\infty$, and we denote this case by $(x^\nu)\to(x)$ or $\lim\,(x^\nu)=(x)$. We will also use the notation $d\left(\frac{x^\nu}{\|x^\nu\|},\frac{x}{\|x\|}\right)\to 0$, as $\nu\to\infty$ (Fenchel, 1953).

Let M be a cone in R^n. A ray (x) is called a *limit ray* of a cone M if there is a sequence of rays of the cone M which are different from (x) and which converges to (x) (Fenchel, 1953).

A *closed cone* or a *closed set of rays* is a cone, which contains all its limit rays. A cone is closed in this sense, if and only if it is closed in the usual topology of R^n. A cone is *open,* if and only if the complementary set of rays is a closed cone. This is equivalent to the following definition: M is open, if and only if for every (x) in M there is an $\varepsilon>0$ such that all rays (y) with $[x,y]<\varepsilon$ are in M. The set of such rays (y) is called an $\varepsilon-$*neighborhood* of (x) (Fenchel, 1953).

If a sequence $\{x^\nu\}$ is convergent to x in the usual topology of R^n, then the sequence $\{(x^\nu)\}$ of rays is convergent to the ray (x). But, the converse of this proposition does not hold in general. This case will be illustrated in the following example.

Example 2.1. Let $M:=\{(x,y): x\geq 0,\ y\geq 0\}\subset\mathrm{R}^2$ be a cone. Define a sequence as follows:

$$x^\nu:=\left(1-\frac{1}{\nu},\frac{1}{\nu}\right).$$

Then the sequence $\{x^\nu\}$ is convergent to $x=(1,0)$. If we consider the sequence $\{(x^\nu)\}$ of rays, then we have $[x^\nu,x]\to 0$ as $\nu\to\infty$. Thus, the sequence $\{(x^\nu)\}$ of rays is convergent to the ray $(x)=((1,0))$.

On the other hand, if we define a sequence $\{x^\nu\}$ as

$$x^\nu:=\begin{cases}\frac{\left(1-\frac{1}{\nu},\frac{1}{\nu}\right)}{\left\|\left(1-\frac{1}{\nu},\frac{1}{\nu}\right)\right\|} & \text{, if } \nu \text{ is an odd integer}\\ 2\frac{\left(1-\frac{1}{\nu},\frac{1}{\nu}\right)}{\left\|\left(1-\frac{1}{\nu},\frac{1}{\nu}\right)\right\|} & \text{, if } \nu \text{ is an even integer}\end{cases},$$

then this sequence is not convergent in the usual topology of R^2. But the sequence $\{(x^\nu)\}$ of rays is convergent to the ray $((1,0))$.

Theorem 2.1. If a sequence $\{(x^\nu)\}$ of rays converges, then its limit is unique.

Theorem 2.2. If the sequence $\{(x^\nu)\}$ converges to (x), then every subsequence of $\{(x^\nu)\}$ also converges to (x).

The proofs of theorems above are similar to those of the ordinary case.

In order to prove the Theorem 3.1, we need to introduce the concept of the addition of two rays. Let $(x),(y)\in(\mathrm{R}^n)$. Then, the *addition* of the rays (x) and (y) is defined by

$$(x)+(y):=\left(\frac{x}{\|x\|}+\frac{y}{\|y\|}\right).$$

If the two rays have opposite direction, then their sum is not a ray. According to the addition operation, the set of rays is not closed. Therefore we add two points on the unit circle. Furthermore, if we apply the addition operation on the circle with radius r, then we obtain the same ray. If we take $(z) := (x) + (y)$, then

$$\begin{aligned}(rz) &= \left(r\frac{x}{\|x\|} + r\frac{y}{\|y\|} \right) \\ &= \left(r\left[\frac{x}{\|x\|} + \frac{y}{\|y\|} \right] \right) \\ &= \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \\ &= (z),\end{aligned}$$

and hence we have $(z) = (rz)$ for all $r > 0$.

We note that the addition of two rays can be defined by using the angles from the positive $x-$axis to the vectors. But, in this paper, we will use the vector sum of two rays.

Since the space of rays is a metric space, we can use the concept of boundedness in a metric space.

Definition 2.1. Let (A) be a set in (R^n). The set (A) is called bounded if there exists a positive integer M and a fixed ray $(y) \in (A)$ such that $[x, y] < M$ for every $(x) \in (A)$.

Since $0 \le [x, y] \le 2$ for all $(x),(y) \in (\mathrm{R}^n)$, every subset (A) of (R^n) is bounded. Furthermore, the set (R^n) is bounded. As a consequence, we can say that every sequence of rays is bounded.

Definition 2.2. Let $(x),(y) \in (\mathrm{R}^2)$. We define $(x) \le (y)$ if and only if $\theta_1 \le \theta_2$, where θ_1 and θ_2 are the angles from the positive $x-$axis to the rays (x) and (y), respectively.

This ordering relation is a linear ordering. We say that the ray which has angle $\theta = 0$ radian is the least ray in (R^2). We denote this ray by (0). But, the greatest ray of (R^2) does not exist.

Unfortunately, we cannot define such an ordering in (R^n) for $n \ge 3$.

The following Definitions 2.3, 2.4 and 2.5 given for the rays are almost the same as the definitions given for real numbers in the book (Stoll, 2001).

Definition 2.3. A subset (A) of (R^2) is bounded above if there exists an $(m) \in (\mathrm{R}^2)$ such that $(x) \le (m)$ for all $(x) \in (A)$. Such an (m) is called an upper bound of (A). Similarly, a subset (A) of (R^2) is bounded below if there exists $(n) \in (\mathrm{R}^2)$ such that $(n) \le (x)$ for all $(x) \in (A)$. Such an (n) is called a lower bound of (A).

We have $(0) \le (x)$ for all $(x) \in (\mathrm{R}^2)$. Namely, (R^2) is bounded below. However, (R^2) is not bounded above with respect to this ordering relation, since there does not exist a $(z) \in (\mathrm{R}^2)$ such that $(x) \le (z)$ for all $(x) \in (\mathrm{R}^2)$.

Definition 2.4. Let (A) be a nonempty subset of (R^2) that is bounded above. An element $(\alpha) \in (\mathrm{R}^2)$ is called the least upper bound or the supremum of (A) if

(i) (α) is an upper bound of (A), and

(ii) if $(\beta) \in (\mathrm{R}^2)$ satisfies $(\beta) < (\alpha)$, then (β) is not an upper bound of (A).

Similarly, let (A) be a nonempty subset of (R^2). An element $(\alpha) \in (\mathrm{R}^2)$ is called the greatest lower bound or the infimum of (A) if

(i) (α) is a lower bound of (A), and

(ii) if $(\beta) \in (\mathrm{R}^2)$ satisfies $(\alpha) < (\beta)$, then (β) is not a lower bound of (A).

Example 2.2. Let

$$(B) := \left\{ (x) \in (\mathrm{R}^2) : \tfrac{\pi}{2} \le \theta \le \pi \right\}.$$

The ray $(\alpha) \in (\mathrm{R}^2)$ with angle $\theta = \frac{\pi}{2}$ is the greatest lower bound of the set (B). Similarly, the ray $(\beta) \in (\mathrm{R}^2)$ with angle $\theta = \pi$ is the least upper bound of the set (B).

Conclusion 2.1. Every nonempty subset of (R^2) that is bounded above has a supremum and an infimum in (R^2).

In order to prove Proposition 3.1, we need to introduce the concepts of the limit superior and the limit inferior of a sequence of rays.

Definition 2.5. Let $\{(x^\nu)\}$ be a sequence of rays in (R^2). The limit superior and the limit inferior of the sequence $\{(x^\nu)\}$ are defined by

$$\overline{\lim_{\nu\to\infty}} (x^\nu) = \inf_{k \in \mathrm{N}} \sup\{(x^\nu) : \nu \ge k\}$$

and

$$\underline{\lim}_{\nu\to\infty} (x^\nu) = \sup_{k \in \mathrm{N}} \inf\{(x^\nu) : \nu \ge k\},$$

respectively.

Theorem 2.3. A sequence $\{(x^\nu)\}$ is convergent to a ray (x) if and only if

$$\underline{\lim}_{\nu\to\infty}(x^{\nu}) = \overline{\lim}_{\nu\to\infty}(x^{\nu}) = (x)$$

Since the proof of the theorem above is similar to the ordinary case, we omit it.

Example 2.3. Let

$$(x^{\nu}) := \begin{cases} ((0,1)) & \text{, if } \nu \text{ is an even integer} \\ ((1,0)) & \text{, if } \nu \text{ is an odd integer} \end{cases}.$$

Then

$$\overline{\lim}_{\nu\to\infty}(x^{\nu}) = ((0,1)) \ \ and \ \ \underline{\lim}_{\nu\to\infty}(x^{\nu}) = ((1,0)).$$

Since

$$\overline{\lim}_{\nu\to\infty}(x^{\nu}) \neq \underline{\lim}_{\nu\to\infty}(x^{\nu})$$

$\lim_{\nu\to\infty}(x^{\nu})$ does not exist.

3. Rough convergence of a sequence of rays

A sequence of rays which is not convergent in the usual sense can be convergent to a ray with certain roughness degree (or certain error).

Definition 3.1. Let $r>0$ be given. A sequence $\{(x^{\nu})\}$ is said to be $r-$convergent to the ray (x) if for every $\varepsilon>0$ there exists a $k(\varepsilon)\in \mathrm{N}$ such that $[x^{\nu},x]<r+\varepsilon$ for all $\nu\geq k(\varepsilon)$, and we denote this situation by $(x^{\nu})\xrightarrow{r}(x)$ or $[x^{\nu},x]\xrightarrow{r}0$. Furthermore, we write $d\left(\frac{x^{\nu}}{\|x^{\nu}\|},\frac{x}{\|x\|}\right)\xrightarrow{r}0$, as $\nu\to\infty$.

The set of all $r-$limit points of $\{(x^{\nu})\}$ is defined by

$$LIM^{r}(x^{\nu}) := \{(x)\in(\mathrm{R}^{n}) : (x^{\nu})\xrightarrow{r}(x)\}.$$

We can say that the set $LIM^{r}(x^{\nu})$ is a cone in R^{n}. For an arbitrary sequence $\{(x^{\nu})\}$, we get $LIM^{r}\left(x^{\nu}\right)=(\mathrm{R}^{n})$ for all $r\geq 2$.

Example 3.1. Let us consider the sequence $\{(x^{\nu})\}$ defined in Example 2.1. Then we have $LIM^{\sqrt{2}}\left(x^{\nu}\right)=\{(x)\in(\mathrm{R}^{2}):\ 0\leq\theta\leq\frac{\pi}{2}$ or $\frac{3\pi}{2}\leq\theta<2\pi\}$.

Since $[x^{\nu},x]<2+\varepsilon$ for all $(x)\in\left(\mathrm{R}^{2}\right)$, we get $LIM^{2}\left(x^{\nu}\right)=(\mathrm{R}^{2})$.

An alternative definition of the rough convergence for a sequence of rays can be given via the following

Proposition 3.1. A sequence $\{(x^{\nu})\}$ is $r-$convergent if and only if

$$\limsup\,[x^{\nu},x]\leq r.$$

Proof. (Necessity) Given $\varepsilon>0$. Let $[x^{\nu},x]<r+\varepsilon$. Then there exists a $k(\varepsilon)\in\mathrm{N}$ such that

$$a^{\nu} := [x^{\nu},x] = \left\|\frac{x^{\nu}}{\|x^{\nu}\|}-\frac{x}{\|x\|}\right\| < r+\varepsilon \text{ for all } \nu\geq k(\varepsilon).$$

Conversely, assume that $\limsup a^{\nu}>r$. Let $\delta=\frac{\limsup a^{\nu}-r}{3}>0$. By definition of the upper limit, we get that, for all $\nu\in\mathrm{N}$ there exists an $m\in\mathrm{N}$ such that

$$m\geq\nu \text{ and } a^{\nu}>\limsup a^{\nu}-\delta.$$

If we choose $\varepsilon=\delta$, then there exists a $k(\varepsilon)\in\mathrm{N}$ such that

$$a^{\nu}<r+\delta \text{ for all } \nu\geq k(\varepsilon),$$

which is a contradiction. Hence we obtain $\limsup\,[x^{\nu},x]\leq r$.

(Sufficiency) Now assume that $\limsup a^{\nu}\leq r$. Suppose also on the contrary that, there exists an $\widetilde{\varepsilon}>0$ such that for all $k(\widetilde{\varepsilon})\in N$ there exists an $m\in\mathrm{N}$ with $m\geq\nu$ and $a^{\nu}\geq r+\widetilde{\varepsilon}$. By hypothesis, we have

$$a^{\nu}<\limsup a^{\nu}+\widetilde{\varepsilon} \ \text{ for all } \ \nu\geq k(\widetilde{\varepsilon}).$$

This inequality contradicts to the fact that $a^{\nu}\geq r+\widetilde{\varepsilon}$ thus the proof is complete. □

Theorem 3.1. If $(x^{\nu})\xrightarrow{r}(x)$ and $(y^{\nu})\xrightarrow{r}(y)$, then we have $(x^{\nu}+y^{\nu})\xrightarrow{2r}(x+y)$, where the terms x^{ν} and y^{ν} as well as x and y are not in opposite direction for all $\nu\in\mathrm{N}$.

Proof. Given $\varepsilon>0$. If $(x^{\nu})\xrightarrow{r}(x)$, then there exists a $k_1(\varepsilon)\in\mathrm{N}$ such that

$$[x^{\nu},x] = \left\|\frac{x^{\nu}}{\|x^{\nu}\|}-\frac{x}{\|x\|}\right\| < r+\frac{\varepsilon}{2} \text{ for all } \nu\geq k_1(\varepsilon).$$

Similarly, if $(y^{\nu})\xrightarrow{r}(y)$, then there exists a $k_2(\varepsilon)\in\mathrm{N}$ such that

$$[y^{\nu},y] = \left\|\frac{y^{\nu}}{\|y^{\nu}\|}-\frac{y}{\|y\|}\right\| < r+\frac{\varepsilon}{2} \text{ for all } \nu\geq k_2(\varepsilon).$$

Take $k_0(\varepsilon):=\max\{k_1(\varepsilon),k_2(\varepsilon)\}$. We have

$$\begin{aligned}[x^\nu + y^\nu, x+y] &= \left\| \left[\frac{x^\nu}{\|x^\nu\|} + \frac{y^\nu}{\|y^\nu\|} \right] - \left[\frac{x}{\|x\|} + \frac{y}{\|y\|} \right] \right\| \\ &= \left\| \left[\frac{x^\nu}{\|x^\nu\|} - \frac{x}{\|x\|} \right] + \left[\frac{y^\nu}{\|y^\nu\|} - \frac{y}{\|y\|} \right] \right\| \\ &\le \left\| \frac{x^\nu}{\|x^\nu\|} - \frac{x}{\|x\|} \right\| + \left\| \frac{y^\nu}{\|y^\nu\|} - \frac{y}{\|y\|} \right\| \\ &< 2r + \varepsilon \end{aligned}$$

for all $\nu \ge k_0(\varepsilon)$. Thus we get $(x^\nu + y^\nu) \overset{2r}{\to} (x+y)$.□

Remark 3.1. Note that, even if one of the sequences defined above is not convergent, the sequence $(x^\nu + y^\nu)$ is convergent to the whole (R^n) for $r \ge 1$.

Theorem 3.2. If $(x^\nu) \overset{r}{\to} (x)$, then $(\lambda x^\nu) \overset{r}{\to} (\lambda x) = (x)$ for all $\lambda > 0$.

Since $(x^\nu) = (\lambda x^\nu)$ and $(x) = (\lambda x)$ for all $\lambda > 0$, the proof of the above theorem is obvious.

In both of the theorems stated above, if we take $r = 0$ then we obtain the classical (non-rough) convergence results.

The proofs of the following theorems are analogues of Phu's Theorems (Phu, 2001).

Theorem 3.3. The cone $LIM^r(x^\nu)$ is closed.

Proof. Let $\varepsilon > 0$ be given. A sequence $\{(y^w)\}$ converges to (y) and $(y^w) \in LIM^r(x^\nu)$ for all $w \in \mathrm{N}$. Then there exists a $k_1(\varepsilon) \in \mathrm{N}$ such that $[y^w, y] < \frac{\varepsilon}{2}$ for all $w \ge k_1(\varepsilon)$. Since $\{(y^w)\} \subset LIM^r(x^\nu)$, we have $(y^{w(\varepsilon)}) \in LIM^r(x^\nu)$. Then there exists a $k_2(\varepsilon) \in \mathrm{N}$ such that $[x^\nu, y^{w(\varepsilon)}] < r + \frac{\varepsilon}{2}$ for all $\nu \ge k_2(\varepsilon)$. Take $\tilde{k}(\varepsilon) := \max\{k_1(\varepsilon), k_2(\varepsilon)\}$. Then

$$\begin{aligned}[x^\nu, y] &\le [x^\nu, y^{w(\varepsilon)}] + [y^{w(\varepsilon)}, y] \\ &< r + \varepsilon\end{aligned}$$

for all $\nu \ge \tilde{k}(\varepsilon)$. Thus we get $(y) \in LIM^r(x^\nu)$.

Theorem 3.4. If $r_1 \le r_2$ then we have $LIM^{r_1}(x^\nu) \subseteq LIM^{r_2}(x^\nu)$.

Proof. Let $r_1 \le r_2$ and $(x) \in LIM^{r_1}(x^\nu)$. Then, for every $\varepsilon > 0$ there exists a $k(\varepsilon) \in \mathrm{N}$ such that

$$\begin{aligned}[x^\nu, x] &< r_1 + \varepsilon \\ &\le r_2 + \varepsilon\end{aligned}$$

for all $\nu \ge k(\varepsilon)$. Thus we have $(x) \in LIM^{r_2}(x^\nu)$.□

Theorem 3.5. If $\{(x^{\nu_k})\}$ is a subsequence of $\{(x^\nu)\}$, then $LIM^r(x^\nu) \subseteq LIM^r(x^{\nu_k})$.

Proof. Let $(x) \in LIM^r(x^\nu)$. Then, for every $\varepsilon > 0$ there exists a $k(\varepsilon) \in \mathrm{N}$ such that $[x^\nu, x] < r + \varepsilon$ for all $\nu \ge k(\varepsilon)$. Since $\{(x^{\nu_k})\}$ is a subsequence of $\{(x^\nu)\}$, we have $[x^{\nu_k}, x] < r + \varepsilon$ for all $\nu_k > k(\varepsilon)$. Thus we have $(x) \in LIM^r(x^{\nu_k})$.□

Definition 3.2. Given a ray $(x) \in (\mathrm{R}^n)$ and a real number $r > 0$, we call a set

$$\overline{B}_r[(x)] := \{(y) \in (\mathrm{R}^n) : [x, y] \le r\}$$

a closed ball in (R^n) with center (x) and radius r. This set is a cone in R^n.

Theorem 3.6. If a sequence $\{(x^\nu)\}$ converges to (x) then $LIM^r(x^\nu) = \overline{B}_r[(x)]$.

Proof. If $(x^\nu) \to (x)$, then for every $\varepsilon > 0$ there exists a $k(\varepsilon) \in \mathrm{N}$ such that $[x^\nu, x] < \varepsilon$ for all $\nu \ge k(\varepsilon)$. Let us choose an arbitrary $(y) \in \overline{B}_r[(x)]$. Since $[x, y] \le r$, we obtain

$$\begin{aligned}[x^\nu, y] &\le [x^\nu, x] + [x, y] \\ &< r + \varepsilon\end{aligned}$$

for all $\nu \ge k(\varepsilon)$. Thus we have $(y) \in LIM^r(x^\nu)$. That is, $\overline{B}_r[(x)] \subset LIM^r(x^\nu)$.

Now let us take a $(y) \in LIM^r(x^\nu)$. Let $\varepsilon > 0$ be arbitrary. Then, there exists a $k_1(\varepsilon) \in \mathrm{N}$ such that $[x^\nu, y] < r + \varepsilon$ for all $\nu \ge k_1(\varepsilon)$. Since $(x^\nu) \to (x)$, there exists a $k_2(\varepsilon) \in \mathrm{N}$ such that $[x^\nu, x] < \varepsilon$ for all $\nu \ge k_2(\varepsilon)$. Now take $k(\varepsilon) := \max\{k_1(\varepsilon), k_2(\varepsilon)\}$. Then

$$\begin{aligned}[x, y] &\le [x, x^\nu] + [x^\nu, y] \\ &\le r + 2\varepsilon\end{aligned}$$

for all $\nu \ge k(\varepsilon)$. Since ε is arbitrary, we have $[x, y] \le r$. Hence we have $(y) \in \overline{B}_r[(x)]$. That is, $LIM^r(x^\nu) \subset \overline{B}_r[(x)]$. Thus the proof is complete. □

Proposition 3.2. (a) If (c) is a cluster point of a sequence $\{(x^\nu)\}$, then $LIM^r(x^\nu) \subseteq \overline{B}_r[(c)]$.

(b) Let (C) be the set of cluster points of a sequence $\{(x^\nu)\}$. Then we have

$$LIM^r(x^\nu) = \bigcap_{(c) \in (C)} \overline{B}_r[(c)] = \{(x) \in (\mathrm{R}^n) : (C) \subseteq \overline{B}_r[(x)]\}.$$

Proof. (a) Assume that the sequence $\{(x^\nu)\}$ has a cluster point (c). Then we get $[x, c] \le r$ for every $(x) \in LIM^r(x^\nu)$. On the other hand, there are infinitely many rays (x^ν) such that $[x^\nu, x] \ge r + \varepsilon$ for $\varepsilon := ([x, c] - r)/3 > 0$, but this inequality contradicts to the fact that

(b) Since $LIM^r(x^\nu) \subseteq \overline{B}_r[(c)]$, we have $LIM^r(x^\nu) = \bigcap_{(c)\in(C)} \overline{B}_r[(c)]$. Let $(y) \in \bigcap_{(c)\in(C)} \overline{B}_r[(c)]$. Thus, we obtain $[y,c] \le r$ for every $(c) \in (C)$. This implies the inclusion $(C) \subseteq \overline{B}_r[(y)]$. That is,

$$\bigcap_{(c)\in(C)} \overline{B}_r[(c)] = \{(x) \in (\mathrm{R}^n) : (C) \subseteq \overline{B}_r[(x)]\}.$$

Now assume that $(y) \notin LIM^r(x^\nu)$. Thus, we can find an $\varepsilon > 0$ such that there exist infinitely many rays (x^ν) such that $[x^\nu, y] \ge r + \varepsilon$. Since $[y,c] \ge r + \varepsilon$, we have a cluster point (c) of $\{(x^\nu)\}$. Hence we have $(C) \not\subseteq \overline{B}_r[(y)]$ and $(y) \notin \{(x) \in (\mathrm{R}^n) : (C) \subseteq \overline{B}_r[(x)]\}$. Since $(y) \in \{(x) \in (\mathrm{R}^n) : (C) \subseteq \overline{B}_r[(x)]\}$, we obtain $(y) \in LIM^r(x^\nu)$. That is,

$$\{(x) \in (\mathrm{R}^n) : (C) \subseteq \overline{B}_r[(x)]\} \subseteq LIM^r(x^\nu).$$

This completes the proof. □

Theorem 3.7. A sequence $\{(x^\nu)\}$ is $r-$convergent to (x) if there exists a sequence $\{(y^\nu)\}$ satisfying $(y^\nu) \to (x)$ and $[x^\nu, y^\nu] \le r$ for all $\nu \in \mathrm{N}$.

Proof. Suppose that $(y^\nu) \to (x)$. Then, for every $\varepsilon > 0$ there exists a $k(\varepsilon) \in \mathrm{N}$ such that $[y^\nu, y] < \varepsilon$ for all $\nu \ge k(\varepsilon)$. Since $[x^\nu, y^\nu] \le r$ for all $\nu \in \mathrm{N}$, we have

$$\begin{aligned}[x^\nu, x] &\le [x^\nu, y^\nu] + [x, y^\nu] \\ &\le r + \varepsilon\end{aligned}$$

for all $\nu \ge k(\varepsilon)$. □

We note that since $0 \le [x, y] \le 2$ for all $(x),(y) \in (\mathrm{R}^n)$, we can write

$$diam(LIM^r(x^\nu)) = \sup\{[x, y] : (x),(y) \in LIM^r(x^\nu)\} \le 2.$$

That is, we say that the diameter of an $r-$limit set is not greater than 2.

Definition 3.3. A subset (A) of (R^n) is said to be convex provided that, for every $(x),(y) \in (A)$ and for all $\lambda \in [0,1]$ we have $\lambda(x) + [1-\lambda](y) = (x) + (y) \in (A)$.

Hence the definitions of convexity of a set in R^n and the set of rays in (R^n) coincide with each other.

Remark 3.2. Although the set $LIM^r x^\nu$ is convex in R^n, the set $LIM^r(x^\nu)$ is not convex in (R^n). This will be illustrated in following example.

Example 3.2. Let $\{(x^\nu)\} = \{((1,1))\} \subset (\mathrm{R}^2)$. Then we have $LIM^{\frac{3}{2}}(x^\nu) = \{(x) \in (\mathrm{R}^2) :\ 0 \le \theta \le \pi \text{ or } \frac{3\pi}{2} \le \theta < 2\pi\}$. This set is not convex in (R^2). In addition, the set $LIM^r(x^\nu)$ is a convex set for $r < 1$ but this set is not convex for $r \ge 1$.

4. Conclusion

In the space of rays, the concepts of metric boundedness and order boundedness are different from each other. Furthermore, the definitions of the convexity of a set in R^n and the convexity of the set of rays in (R^n) coincide with each other. Although the set $LIM^r x^\nu$ is convex in R^n, the set $LIM^r(x^\nu)$ is not convex in (R^n). In addition, if $x^\nu \to x$ and $y^\nu \to y$ then $x^\nu + y^\nu \to x + y$ in classical analysis. But, if $(x^\nu) \xrightarrow{r} (x)$ and $(y^\nu) \xrightarrow{r} (y)$ then $(x^\nu + y^\nu) \xrightarrow{2r} (x + y)$ in the setting of rough convergence of a sequence of rays.

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خلاصة

نقوم في هذا العمل بإدخال مفهوم التقارب التقريبي لمتتالية أشعة و نحصل على بعض النتائج الأساسية . وفي هذا السياق ، إذا أخذنا r=o ، فأننا نحصل على النتائج الكلاسيكية لنظرية الأشعة.