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Description of Bloch spaces, weighted Bergman spaces and invariant subspaces, and related questions

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Abstract

Let $\mathbb{D}$ be the unit disc of complex plane $\mathbb{C}$, and $\mathbf{H} = Hol(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. Recall that an $f \in Hol(\mathbb{D})$ is said to belong to the Bloch space $\mathbf{B} = \mathbf{B}(\mathbb{D})$ if $\|f\|_{\mathbf{B}} := \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < +\infty$. With the norm $\|f\| = |f(0)| + \|f\|_{\mathbf{B}}$, $\mathbf{B}$ is Banach space. Let $\mathbf{B}_0 = \mathbf{B}_0(\mathbb{D})$ be the Bloch space which consists of all $f \in \mathbf{B}$ satisfying $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. Here we give a new description of Bloch spaces and weighted Bergman spaces in terms of Berezin symbols of diagonal operators associated with the Taylor coefficients of their functions. We also give in terms of Berezin symbols a characterization of the multiple shift invariant subspaces of weighted Bergman spaces. Some other questions are also discussed.

Keywords: Berezin symbol; Bloch space; diagonal operator; invariant subspace; weighted Bergman space.

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc, and $Hol(\mathbb{D})$ the class of functions halomorphic (analytic) in $\mathbb{D}$. We denote by $H^\infty = H^\infty(\mathbb{D})$ the bounded analytic functions space on $\mathbb{D}$. The Bloch space $\mathbf{B}$ of $\mathbb{D}$ is defined to be the space of analytic functions *f* on $\mathbb{D}$ such that

$$\|f\|_{\mathbf{B}} := \sup\{(1-|z|^2)|f'(z)| : z \in \mathbb{D}\} < +\infty,$$

$\|\cdot\|_{\mathbf{B}}$ is a complete semi-norm on $\mathbf{B}$. $\mathbf{B}$ can made into a Banach space by introducing the norm

$$\|f\| = |f(0)| + \|f\|_{\mathbf{B}}.$$

The little Bloch space of $\mathbb{D}$, denoted $\mathbf{B}_0$, is the closed subspace of $\mathbf{B}$ consisting of functions *f* with

$$(1-|z|^2)f'(z) \to 0 \ (|z| \to 1^-).$$

It is the closure of the set of polynomials in $\mathbf{B}$.

Let $C(\overline{\mathbb{D}})$ be the space of continuous functions on the closed unit disc $\overline{\mathbb{D}}$. Denote by $C_0(\mathbb{D})$ the subspace of $C(\overline{\mathbb{D}})$ consisting of functions vanishing on the unit circle $\mathbb{T}$. Clearly, both $C(\overline{\mathbb{D}})$ and $C_0(\mathbb{D})$ are closed subspaces of $L^\infty(\mathbb{D})$.

We set

$$Hol(\mathbb{D}) = \{f : f \text{ is analytic on } \mathbb{D}\}.$$

In this article we give a new characterization of Bloch spaces and some weighted Bergman spaces A_α^p in terms of Berezin symbols of diagonal operators associated with the Taylor coefficients of functions in these spaces. A new characterization of A_α^p-interpolating and sampling sequences is given in terms of Berezin symbols. We also characterize in terms of Berezin symbols the multiple shift-invariant subspaces of weighted Bergman spaces. For more detailed and different characterizations of Bloch spaces and weighted Bergman spaces, the reader can refer to Zhu (2007) and Hedenmalm *et al.* (2000). A characterization of some other function classes is contained in Karaev (2012).

2. Diagonal operators and characterization of Bloch spaces

In the present section, we characterize Bloch spaces in terms of Berezin symbols.

Recall that for a bounded linear operator *A*, acting on the Hardy-Hilbert space $H^2 = H^2(\mathbb{D})$ of analytic functions

$f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ with the sequence of Taylor coefficients $\left(\hat{f}(n)\right)_{n\geq 0}=\left(\frac{f^{(n)}(0)}{n!}\right)_{n\geq 0}$ in ℓ^2, its Berezin symbol $\widetilde{A}$ is defined by

$$\widetilde{A}(\lambda)=\left\langle A\hat{k}_\lambda(z),\hat{k}_\lambda(z)\right\rangle\ (\lambda\in\mathbb{D}),$$

where $\hat{k}_\lambda(z)=\frac{k_\lambda(z)}{\|k_\lambda(z)\|_{H^2}}=\frac{\left(1-|\lambda|^2\right)^{1/2}}{1-\bar{\lambda}z}$ is the normalized reproducing kernel of the Hardy space H^2. Clearly, $\left|\widetilde{A}(\lambda)\right|\leq\|A\|$, and hence $\widetilde{A}$ is a bounded function on $\mathbb{D}$.

For any bounded sequence $\{a_n\}_{n\geq 0}$ of complex numbers a_n, let $D_{(a_n)}$ denote the associated diagonal operator on the Hardy space H^2 defined by

$$D_{(a_n)}z^k=a_kz^k,\ k=0,1,2,.... \tag{1}$$

Then, it can be easily shown that (Ash & Karaev, 2012)

$$\widetilde{D}_{(a_n)}(\lambda)=\left(1-|\lambda|^2\right)\sum_{n=0}^{\infty}a_n|\lambda|^{2n}\ (\lambda\in\mathbb{D}), \tag{2}$$

which means that Berezin symbol $\widetilde{D}_{(a_n)}$ of the diagonal operator $D_{(a_n)}$ defined by the formula (1) is a radial function, that is $\widetilde{D}_{(a_n)}(\lambda)=\widetilde{D}_{(a_n)}(|\lambda|)$ for every $\lambda\in\mathbb{D}$.

The following two results characterize the Bloch spaces functions in terms of their Taylor coefficients, which is one of important and classical questions of the theory of analytic functions (Privalov, 1950; Duren, 2000).

Theorem 2.1. Let $f(z)=\sum_{k=0}^{\infty}\hat{f}(k)z^k\in Hol(\mathbb{D})$ be a function with the sequence $\left(\hat{f}(k)\right)_{k\geq 0}$ of Taylor coefficients $\hat{f}(k)=f^{(k)}(0)/k!$ $(k=0,1,2,...)$, satisfying $\hat{f}(k+n)=O\left(\frac{k!}{(k+n)!}\right)$ as $k\to+\infty$ for any $n\geq 2$. Then $f\in\mathbf{B}$ if and only if

$$\sup_{\substack{0\leq r<1\\ 0\leq\theta<2\pi}}(1-r)^{n-1}\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|<+\infty.$$

Proof. Let $f\in\mathbf{B}$ and $n\geq 1$. Then we have

$$f^{(n)}(z)=\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\hat{f}(k+n)z^k. \tag{3}$$

By considering that $z=|z|e^{i\arg(z)}=:re^{i\theta}$, where $r=|z|$, $\theta=\arg(z)$, and hence $0\leq r<1$ and $0\leq\theta<2\pi$, we have for $n\geq 2$ from the representation (3) that

$$\begin{aligned}&\left(1-|z|^2\right)^n\left|f^{(n)}(z)\right|\\&=\left(1-r^2\right)^n\left|\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}r^k\right|\\&=(1+r)^n(1-r)^{n-1}\left|(1-r)\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}r^k\right|.\end{aligned}$$

By considering the condition of the theorem and formula (2), we obtain from the latter that

$$\begin{aligned}&\left(1-|z|^2\right)^n\left|f^{(n)}(z)\right|\\&=(1+r)^n(1-r)^{n-1}\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|\end{aligned} \tag{4}$$

for all $r\in[0,1)$ and $\theta\in[0,2\pi)$. Since $1\leq(1+r)^n<2^n$, we conclude from the latter that

$$\begin{aligned}&\sup_{\substack{0\leq r<1\\ 0\leq\theta<2\pi}}(1-r)^{n-1}\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|\\&\leq\sup_{\substack{0\leq r<1\\ 0\leq\theta<2\pi}}\left(1-|z|^2\right)^n\left|f^{(n)}(z)\right|\\&\leq 2^n\sup_{\substack{0\leq r<1\\ 0\leq\theta<2\pi}}(1-r)^{n-1}\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|,\end{aligned}$$

or equivalently, $\left(1-|z|^2\right)^nf^{(n)}(z)$ is bounded in $\mathbb{D}$ if and only if

$$\sup_{\substack{0\leq r<1\\ 0\leq\theta<2\pi}}(1-r)^{n-1}\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|<+\infty.$$

For completing the proof, now it remains only to apply the well-known fact that (Zhu, 2007 [Theorem 5.1.5]) if f is analytic in $\mathbb{D}$ and $n\geq 2$, then $f\in\mathbf{B}$ if and only if $\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^n\left|f^{(n)}(z)\right|<+\infty$. This proves the theorem.

Theorem 2.2. Let $f(z)=\sum_{k=0}^{\infty}\hat{f}(k)z^k\in Hol(\mathbb{D})$ be a function and n is a positive integer such that $\hat{f}(k+n)=O\left(\frac{k!}{(k+n)!}\right)$ as $k\to+\infty$. Then $f\in\mathbf{B}_0$ if and only if

$$\lim_{r\to 1^-}(1-r)^{n-1}\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)=0$$

for all $\theta\in[0,2\pi)$.

Proof. The proof is very similar to the proof of Theorem 2.1. Indeed, as in the proof of Theorem 2.1, we obtain that

$$\begin{aligned}&\left(1-|z|^2\right)^nf^{(n)}(z)\\&=(1+r)^n(1-r)^{n-1}\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\end{aligned} \tag{5}$$

for any $r \in [0,1)$ and $\theta \in [0,2\pi)$. Since by condition of the theorem $\left|\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right|$ is bounded for any fixed $\theta \in [0,2\pi)$, we conclude that the diagonal operator $D_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}$ is bounded for any fixed $\theta \in [0,2\pi)$ and hence its Berezin symbol $\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}$ is bounded on $\mathbb{D}$. Now it remains only to use (5) and the known facts that (Hedenmalm *et al.*, 2000 [Proposition 1.13]) if n is a positive integer and f is analytic in $\mathbb{D}$, then $f \in \mathbf{B}$ if and only if $\left(1-|z|^2\right)^n f^{(n)}(z)$ is in $L^\infty(\mathbb{D})$, and $f \in \mathbf{B}_0$ if and only if the function $\left(1-|z|^2\right)^n f^{(n)}(z)$ is in $C(\overline{\mathbb{D}})$ (or $C_0(\mathbb{D})$).

Recall that a sequence $(\lambda_n)_n$ of positive integers is called a *gap* sequence if there exists a constant $\lambda > 1$ such that $\lambda_{n+1}/\lambda_n \ge \lambda$ for all $n = 1,2,3,\ldots$. In this case, we call a power series of the form $\sum_{n=0}^{\infty} a_n z^{\lambda_n}$ a *lacunary series*.

Theorem 2.3. Let $f(z) = \sum_{n=0}^{+\infty} a_n z^{\lambda_n}$ be a lacunary power series. Then

(a) $f \in \mathbf{B}$ if and only if

$$\sup_{\substack{0<r<1 \\ 0\le\theta<2\pi}} \left|\widetilde{D}_{\left(a_n\lambda_n e^{i(\lambda_n-1)\theta} r^{\lambda_n-n-1}\right)}\left(\sqrt{r}\right)\right| < +\infty.$$

(b) $f \in \mathbf{B}_0$ if and only if

$$\lim_{r\to 1^-} \left|\widetilde{D}_{\left(a_n\lambda_n e^{i(\lambda_n-1)\theta} r^{\lambda_n-n-1}\right)}\left(\sqrt{r}\right)\right| = 0$$

for all $\theta \in [0,2\pi)$.

Proof. In fact, $f\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} a_n \left(re^{i\theta}\right)^{\lambda_n}$, and therefore

$$\begin{aligned} f'\left(re^{i\theta}\right) &= \sum_{n=0}^{\infty} a_n\lambda_n \left(re^{i\theta}\right)^{\lambda_n-1} \\ &= \sum_{n=0}^{\infty} \left(a_n\lambda_n e^{i(\lambda_n-1)\theta} \frac{r^{\lambda_n-1}}{r^n}\right) r^n \\ &= \sum_{n=0}^{\infty} \left(a_n\lambda_n e^{i(\lambda_n-1)\theta} r^{\lambda_n-n-1}\right) r^n. \end{aligned}$$

Then we have

$$\begin{aligned} \left(1-|z|^2\right)|f'(z)| &= \left(1-r^2\right)\left|\sum_{n=0}^{\infty}\left(a_n\lambda_n e^{i(\lambda_n-1)\theta} r^{\lambda_n-n-1}\right) r^n\right| \\ &= (1+r)\left|(1-r)\sum_{n=0}^{\infty}\left(a_n\lambda_n e^{i(\lambda_n-1)\theta} r^{\lambda_n-n-1}\right) r^n\right| \\ &= (1+r)\left|\widetilde{D}_{\left(a_n\lambda_n e^{i(\lambda_n-1)\theta} r^{\lambda_n-n-1}\right)}\left(\sqrt{r}\right)\right|, \end{aligned}$$

thus

$$\begin{aligned} &\left|\widetilde{D}_{\left(a_n\lambda_n e^{i(\lambda_n-1)\theta} r^{\lambda_n-n-1}\right)}\left(\sqrt{r}\right)\right| \\ &\le \left(1-|z|^2\right)|f'(z)| \le 2\left|\widetilde{D}_{\left(a_n\lambda_n e^{i(\lambda_n-1)\theta} r^{\lambda_n-n-1}\right)}\left(\sqrt{r}\right)\right|, \end{aligned} \tag{6}$$

for all $\theta \in [0,2\pi)$, which implies that

$$\sup_{z\in\mathbb{D}} \left(1-|z|^2\right)|f'(z)| < +\infty$$

if and only if

$$\sup_{\substack{0<r<1 \\ 0\le\theta<2\pi}} \left|\widetilde{D}_{\left(a_n\lambda_n e^{i(\lambda_n-1)\theta} r^{\lambda_n-n-1}\right)}\left(\sqrt{r}\right)\right| < +\infty,$$

which proves (a).

Inequalities (6) also easily imply (b), which proves the theorem.

Corollary 1. (a) If $a_n = O\left(\frac{1}{\lambda_n r^{\lambda_n-n-1}}\right)$ $(n\to\infty)$ for all $r \in (0,1)$, then the lacunary series $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ defines a function in $\mathbf{B}$.

(b) If $a_n = o\left(\frac{1}{\lambda_n r^{\lambda_n-n-1}}\right)$ $(n\to\infty)$ for all $r \in (0,1)$, then the lacunary series $f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$ defines a function in $\mathbf{B}_0$.

Proof. Since $\left|\widetilde{D}_{(b_n)}(\lambda)\right| \le \sup_{n\ge 0}|b_n|$ for any bounded diagonal operator $D_{(b_n)}$ on H^2, the proof of (a) is trivial.

On the other hand, since $D_{(b_n)}$ is a compact diagonal operator on H^2 if and only if $b_n \to 0$ $(n\to\infty)$, and for the compact operator $D_{(b_n)}$, $\widetilde{D}_{(b_n)}$ vanishes on the boundary $\mathbb{T} = \partial\mathbb{D}$, we get (b). This proves the corollary.

3. A description of weighted Bergman spaces

The normalized area measure on $\mathbb{D}$ will be denoted by dA. In terms of real (rectangular and polar) coordinates, we have

$$dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta,\ z = x+iy = re^{i\theta}.$$

For $0 < p < +\infty$ and $-1 < \alpha < +\infty$, the (weighted) Bergman space $A_\alpha^p = A_\alpha^p(\mathbb{D})$ of the disc is the space of analytic functions in $L^p(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) := (\alpha+1)\left(1-|z|^2\right)^\alpha dA(z).$$

If f is in $L^p(\mathbb{D}, dA_\alpha)$, we write

$$\|f\|_{p,\alpha} = \left[\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z)\right]^{1/p}.$$

When $1 \le p < +\infty$, the space $L^p(\mathbb{D}, dA_\alpha)$ is a Banach space with the above norm; when $0<p<1$, the space $L^p(\mathbb{D},dA_\alpha)$ is a complete metric space with the metric defined by

$$d(f,g)=\|f-g\|_{p,\alpha}^p.$$

We let $L^\infty(\mathbb{D})$ denote the space of (essentially) bounded functions on $\mathbb{D}$. For $f\in L^\infty(\mathbb{D})$ we define

$$\|f\|_\infty = ess\sup\{|f(z)| \ : \ z\in\mathbb{D}\}.$$

The space $L^\infty(\mathbb{D})$ is a Banach space with the above norm. For more informations about weighted Bergman spaces A_α^p, see, for instance, Hedenmalm *et al.* (2000).

Our next result characterizes some weighted Bergman spaces in terms of Berezin symbols of diagonal operator corresponding to the Taylor coefficients of functions in these spaces.

Theorem 3.4. Suppose $1\le p<+\infty$, $-1<\alpha<+\infty$, and that n is a positive integer. Then an analytic function f in $\mathbb{D}$ belongs to A_α^p if and only if

$$\int_0^{2\pi}\int_0^1 r(1-r)^{(n-1)p+\alpha}\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|^p drd\theta<+\infty.$$

Proof. Let $f(z)=\sum_{k=0}^\infty \hat{f}(k)z^k$ be an analytic function in $\mathbb{D}$. Then by using formula (4), we obtain:

$$\left|\left(1-|z|^2\right)^n f^{(n)}(z)\right|^p$$
$$=(1+r)^{np}(1-r)^{(n-1)p}\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|^p,$$

and hence

$$\int_{\mathbb{D}}\left(1-|z|^2\right)^{np}\left|f^{(n)}(z)\right|^p dA_\alpha(z)$$
$$=\int_0^{2\pi}\int_0^1(1+r)^{np}(1-r)^{(n-1)p}\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|^p.$$
$$.(\alpha+1)\left(1-r^2\right)^\alpha\frac{rdrd\theta}{\pi}$$
$$=\frac{\alpha+1}{\pi}\int_0^{2\pi}\int_0^1 r(1+r)^{np+\alpha}(1-r)^{(n-1)p+\alpha}.$$
$$.\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|^p drd\theta$$
$$\le\frac{\alpha+1}{\pi}2^{np+\alpha}\int_0^{2\pi}\int_0^1 r(1-r)^{(n-1)p+\alpha}.$$
$$.\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|^p drd\theta.$$

On the other hand, since $1+r\ge 1$, we have

$$\int_{\mathbb{D}}\left(1-|z|^2\right)^{np}\left|f^{(n)}(z)\right|^p dA_\alpha(z)$$
$$\ge\frac{\alpha+1}{\pi}\int_0^{2\pi}\int_0^1 r(1-r)^{(n-1)p+\alpha}.$$
$$.\left|\widetilde{D}_{\left(\frac{(k+n)!}{k!}\hat{f}(k+n)e^{ik\theta}\right)}\left(\sqrt{r}\right)\right|^p drd\theta.$$

The last two inequalities together with Proposition 1.11 in Hedenmalm *et al.* (2000), give the proof of the theorem.

The above used method of diagonal operators can also be useful to study the zero sequences of some analytic functions on $\mathbb{D}$.

Proposition 3.1. Let $f(z)=\sum_{n=0}^\infty \hat{f}(n)z^n$ be an analytic function on $\mathbb{D}$ with the bounded sequence of Taylor coefficients. If $(\lambda_k)_{k\ge0}$ is the zero sequence of f, then

$$\lim_{t\to1^-}\frac{\widetilde{D}_{\left(\hat{f}(n)\lambda_k^n\right)}\left(\sqrt{t}\right)}{1-t}=0 \tag{7}$$

for any $k\ge0$.

Proof. Since $f(\lambda_k)=0$, $k=0,1,2,...$, it means that $0=f(\lambda_k)=\sum_{n=0}^\infty\hat{f}(n)\lambda_k^n$ for any $k\ge0$, that is the numerical series $\sum_{n=0}^\infty\hat{f}(n)\lambda_k^n$ converges to 0 for any $k\ge0$. Then it converges to 0 also in Abel sense, which means that $\sum_{n=0}^\infty\left(\hat{f}(n)\lambda_k^n\right)t^n$ is convergent for all $t\in(0,1)$ and $\lim_{t\to1^-}\sum_{n=0}^\infty\left(\hat{f}(n)\lambda_k^n\right)t^n=0$. Therefore

$$\lim_{t\to1^-}\frac{(1-t)\sum_{n=0}^\infty\left(\hat{f}(n)\lambda_k^n\right)t^n}{1-t}=0$$

for any $k\ge0$, and hence it follows from (2) that

$$\lim_{t\to1^-}\frac{\widetilde{D}_{\left(\hat{f}(n)\lambda_k^n\right)}\left(\sqrt{t}\right)}{1-t}=0$$

for any $k\ge0$. This proves the proposition.

Remark 3.1. (a) Since $\lim_{n\to\infty}\hat{f}(n)\lambda_k^n=0\ (\forall k\ge0)$, we have that $\lim_{t\to1^-}\widetilde{D}_{\left(\hat{f}(n)\lambda_k^n\right)}\left(\sqrt{t}\right)=0$. So, condition (7) means some growth condition for the boundary values of the Berezin symbols $\widetilde{D}_{\left(\hat{f}(n)\lambda_k^n\right)}$ of compact operators $D_{\left(\hat{f}(n)\lambda_k^n\right)}$, $k=0,1,2,...$.

(b) If $\left(\hat{f}(n)\right)_{n\geq 0}$ in addition satisfies the Tauberian type condition that $\hat{f}(n)=O\left(\frac{1}{n}\right)$, $n\to\infty$, then the necessary condition (7) is also sufficient in order to a sequence $(\lambda_k)_{k\geq 0}$ be a zero sequence of *f*.

(c) Proposition 3.1, apparently, may also be useful in investigation of the zeros of the Riemann ξ-function.

4. Description of multiple shift invariant subspaces

In this short section, we use the diagonal operators technique in description of invariant subspaces of multiple shift operator $M_{z^n}f=z^nf$ on the weighted Bergman spaces.

Theorem 4.5. Let $E\subset A_\alpha^p$, $1\leq p<+\infty$, $-1<\alpha<+\infty$, be a nonzero proper subspace, and $n\geq 1$ be any integer. Then $M_{z^n}E\subset E$ if and only if for every $f\in E$ there exists a function $g=g_f\in E$ such that

$$\widetilde{D}_{\left(\hat{f}(k-n)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right)=\left(1-|z|\right)g(z),\ z\in\mathbb{D}. \tag{8}$$

Proof. For any $f\in E$ we have:

$$z^nf(z)=z^n\sum_{k=0}^{\infty}\hat{f}(k)z^k=\sum_{k=0}^{\infty}\hat{f}(k)z^{k+n}$$
$$=\sum_{k=n}^{\infty}\hat{f}(k-n)z^k$$
$$=\sum_{k=n}^{\infty}\hat{f}(k-n)e^{ik\arg(z)}|z|^k$$
$$=\frac{\left(1-|z|\right)\sum_{k=0}^{\infty}\hat{f}(k-n)e^{ik\arg(z)}|z|^k}{1-|z|}$$

(we put $\hat{f}(-n)=\hat{f}(-(n-1))=...=\hat{f}(-1):=0$). So, it follows from formula (2) that

$$z^nf(z)=\frac{\widetilde{D}_{\left(\hat{f}(k-n)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right)}{1-|z|},\ z\in\mathbb{D}. \tag{9}$$

Now formula (9) implies that $z^nf\in E$ if and only if

$$\frac{\widetilde{D}_{\left(\hat{f}(k-n)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right)}{1-|z|}\in E,$$

which means that

$$\widetilde{D}_{\left(\hat{f}(k-n)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right)=\left(1-|z|\right)g(z)$$

for some $g=g_f\in E$. Thus, $z^nE\subset E$ if and only if for any $f\in E$ there is a function $g=g_f\in E$ satisfying (8), as desired.

Recall that the class $\ell_A^\infty=\ell_A^\infty(\mathbb{D})$ consists of analytic functions $f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$ on $\mathbb{D}$ with $\left(\hat{f}(n)\right)_{n\geq 0}\in\ell^\infty$. Recall also that the range of the Berezin symbol $\widetilde{A}$ is called the Berezin set of the operator A, and will be denoted as $Ber(A)$, i.e., $Ber(A)=\left\{\widetilde{A}(\lambda):\lambda\in\mathbb{D}\right\}$. Clearly, $Ber(A)\subset W(A)$, where $W(A)$ is the numerical range of operator A defined as $W(A)=\left\{\langle Af,f\rangle : \|f\|_{H^2}=1\right\}$. For any subspace $E\subset\ell_A^\infty$, we set

$$B_E:=\bigcup_{\substack{f\in E\\ \theta\in[0,2\pi)}}Ber\left(\widetilde{D}_{\left(\hat{f}(n-1)e^{in\theta}\right)}\right)$$

and

$$B_{zE}:=\bigcup_{\substack{g\in zE\\ \eta\in[0,2\pi)}}Ber\left(\widetilde{D}_{\left(\hat{g}(n)e^{in\eta}\right)}\right).$$

The following result belongs to Ash & Karaev (2012); below S denotes the usual shift operator defined by $Sf=zf$.

Lemma 4.1. If $f\in\ell_A^\infty$, then $Range\left(\left(1-|z|\right)f\right)=\bigcup_{\theta\in[0,2\pi)}Ber\left(\widetilde{D}_{\left(\hat{f}(n)e^{in\theta}\right)}\right)$.

Our next result describes z-invariant (shift-invariant) subspaces in terms of the sets B_E and B_{zE}.

Theorem 4.6. Let $E\subset A_\alpha^p$ be a nontrivial (closed) subspace (i.e. $\{0\}\neq E\neq X$). Then $SE\subset E$ if and only if $B_E\subseteq B_{zE}$.

Proof. It follows from Theorem 4.5 for $n=1$ that $SE\subset E$ if and only if for any $f\in E$ there exits a function $g=g_f\in E$ such that

$$\widetilde{D}_{\left(\hat{f}(k-1)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right)=\left(1-|z|\right)g(z)\ (\forall z\in\mathbb{D}). \tag{10}$$

By Lemma 4.1, we have then that

$$Range\left(1-|z|\,g(z)\right)=\bigcup_{\theta\in[0,2\pi)}Ber\left(D_{\left(\hat{g}(n)e^{in\theta}\right)}\right).$$

So, for any $z\in\mathbb{D}$ there exists $w=w_z\in\mathbb{D}$ such that

$$\left(1-|z|\right)g(z)=\widetilde{D}_{\left(\hat{g}(n)e^{in\arg(w)}\right)}\left(\sqrt{|w|}\right). \tag{11}$$

On the other hand, since $g\in zE$, we have $\hat{g}(0)=0$ and $zf(z)=\sum_{k\geq 0}\hat{f}(k)z^{k+1}=g(z)=\sum_{k\geq 0}\hat{g}(k)z^k$, and hence $\hat{f}(k-1)=\hat{g}(k)$ for any $k\geq 1$. Then, by considering this and formulas (10) and (11), we obtain

$$\widetilde{D}_{\left(\hat{f}(k-n)e^{ik\arg(z)}\right)}\left(\sqrt{|z|}\right)=\widetilde{D}_{\left(\hat{g}(k)e^{ik\arg(w)}\right)}\left(\sqrt{|w|}\right).$$

Or, equivalently, $SE\subset E$ if and only if $\forall f\in E$, $\exists g\in zE$ such that

$$Ber\left(D_{\left(\hat{f}(k-1)e^{ik\theta}\right)}\right)\subseteq Ber\left(D_{\left(\hat{g}(k)e^{ik\eta}\right)}\right)$$

for all $\theta\in[0,2\pi)$. This means that $SE\subset E$ if and only if

$$\bigcup_{\substack{f\in E\\ \theta\in[0,2\pi)}} Ber\left(D_{\left(\hat{f}(k-1)e^{ik\theta}\right)}\right)\subseteq \bigcup_{\substack{g\in zE\\ \eta\in[0,2\pi)}} Ber\left(D_{\left(\hat{g}(k)e^{ik\eta}\right)}\right),$$

or $B_E\subseteq B_{zE}$. This proves the theorem.

5. Remarks on interpolation and sampling in A_α^p

In this brief section, we show how the technique of diagonal operators of the previous sections can be adapted in study of interpolating and sampling sequences for the weighted Bergman spaces A_α^p.

Recall that a sequence $\Gamma=(z_j)_j$ of distinct points in $\mathbb{D}$ is called an A_α^p-*interpolation sequence* (or a sequence of interpolation for A_α^p) if for every sequence $(w_j)_j$ complex numbers satisfying the condition

$$\sum_j\left(1-|z_j|^2\right)^{2+\alpha}|w_j|^p<+\infty,$$

there exists a function $f\in A_\alpha^p$ such that $f(z_j)=w_j$ for all j (Hedenmalm *et al.*, 2000). Then we have:

$$w_j=f(z_j)=\sum_{n=0}^{\infty}\hat{f}(n)z_j^n=\sum_{n=0}^{\infty}\hat{f}(n)e^{in\arg(z_j)}|z_j|^n$$

$$=\frac{(1-|z_j|)\sum_{n=0}^{\infty}\hat{f}(n)e^{in\arg(z_j)}|z_j|^n}{1-|z_j|}$$

$$=\frac{\widetilde{D}_{\left(\hat{f}(n)e^{in\arg(z_j)}\right)}\left(\sqrt{|z_j|}\right)}{1-|z_j|}.$$

Thus

$$w_j=\frac{\widetilde{D}_{\left(\hat{f}(n)e^{in\arg(z_j)}\right)}\left(\sqrt{|z_j|}\right)}{1-|z_j|},$$

and hence

$$(1-|z_j|)w_j=\widetilde{D}_{\left(\hat{f}(n)e^{in\arg(z_j)}\right)}\left(\sqrt{|z_j|}\right)$$

for all j. Therefore

$$\left(1-|z_j|^2\right)^{2+\alpha}|w_j|^p$$
$$=(1+|z_j|)\left(1-|z_j|^2\right)^{1+\alpha}|w_j|^{p-1}\left|\widetilde{D}_{\left(\hat{f}(n)e^{in\arg(z_j)}\right)}\left(\sqrt{|z_j|}\right)\right|.$$

Since $1\le 1+z_j<2$, the latter implies the proof of the following.

Proposition 5.2. A sequence $\Gamma=(z_j)_j$ of distinct points in $\mathbb{D}$ is an A_α^p-interpolation sequence if and only if

$$\sum_j\left(1-|z_j|^2\right)^{1+\alpha}|f(z_j)|^{p-1}\left|\widetilde{D}_{\left(\hat{f}(n)e^{in\arg(z_j)}\right)}\left(\sqrt{|z_j|}\right)\right|<+\infty$$

for every function f in A_α^p.

Recall that a sequence $\Gamma=(z_j)_j$ of (not necessarily distinct) points in $\mathbb{D}$ is called an A_α^p-*sampling sequence* (or a sequence of sampling for A_α^p) if there exists a positive constant C such that

$$C^{-1}\int_{\mathbb{D}}|f(z)|^p\,dA_\alpha(z)\le\sum_{j=1}^{\infty}\left(1-|z_j|^2\right)^{2+\alpha}|f(z_j)|^p\le C\int_{\mathbb{D}}|f(z)|^p\,dA_\alpha(z)\tag{12}$$

for all $f\in A_\alpha^p$ (Hedenmalm *et al.*, 2000).

Proposition 5.3. A sequence $\Gamma=(z_j)_j$ of (not necessarily distinct) points in $\mathbb{D}$ is an A_α^p-sampling sequence if and only if there exists a positive constant C such that

$$C^{-1}(\alpha+1)\int_0^{2\pi}\int_0^1\left|\widetilde{D}_{\left(\hat{f}(n)e^{in\theta}\right)}(\sqrt{r})\right|^p(1-r)^{\alpha-p}r(1+r)^\alpha\,drd\theta$$
$$\le\sum_{j=1}^{\infty}(1-|z_j|)^{2+\alpha-p}(1+|z_j|)^{2+\alpha}\left|\widetilde{D}_{\left(\hat{f}(n)e^{in\arg(z_j)}\right)}\left(\sqrt{|z_j|}\right)\right|^p$$
$$\le C(\alpha+1)\int_0^{2\pi}\int_0^1\left|\widetilde{D}_{\left(\hat{f}(n)e^{in\theta}\right)}(\sqrt{r})\right|^p(1-r)^{\alpha-p}r(1+r)^\alpha\,drd\theta.$$

Proof. Let $f\in A_\alpha^p$ be arbitrary. Then by using our previous calculus, we have

$$f(z)=\frac{\widetilde{D}_{\left(\hat{f}(n)e^{in\arg(z)}\right)}\left(\sqrt{|z|}\right)}{1-|z|}\ (z\in\mathbb{D}).\tag{13}$$

From (13), we obtain

$$\int_{\mathbb{D}}|f(z)|^p\,dA_\alpha(z)$$
$$=(\alpha+1)\int_{\mathbb{D}}|f(z)|^p\left(1-|z|^2\right)^\alpha dA(z)$$
$$=(\alpha+1)\int_0^{2\pi}\int_0^1\frac{\left|\widetilde{D}_{\left(\hat{f}(n)e^{in\theta}\right)}(\sqrt{r})\right|^p}{(1-r)^p}\left(1-r^2\right)^\alpha rdrd\theta$$
$$=(\alpha+1)\int_0^{2\pi}\int_0^1\left|\widetilde{D}_{\left(\hat{f}(n)e^{in\theta}\right)}(\sqrt{r})\right|^p(1-r)^{\alpha-p}r(1+r)^\alpha\,drd\theta,$$

and also

$$\sum_{j=1}^{\infty}\left(1-\left|z_j\right|^2\right)^{2+\alpha}\left|f\left(z_j\right)\right|^p$$

$$=\sum_{j=1}^{\infty}\left(1-\left|z_j\right|^2\right)^{2+\alpha}\left|\frac{\widetilde{D}_{\left(\hat{f}(n)e^{in\arg\left(z_j\right)}\right)}\left(\sqrt{\left|z_j\right|}\right)}{1-\left|z_j\right|}\right|^p.$$

Since again $1\le(1+r)^{\alpha}<2^{\alpha}$ and $1\le(1+r)^{2+\alpha}<2^{2+\alpha}$, by considering all these we have from the definition of A_{α}^{p}-sampling sequence that a sequence $\Gamma=\left(z_j\right)_j$ is a A_{α}^{p}-sampling sequence if and only if there exists a positive constant C such that

$$C^{-1}(\alpha+1)\int_0^{2\pi}\int_0^1\left|\widetilde{D}_{\left(\hat{f}(n)e^{in\theta}\right)}\left(\sqrt{r}\right)\right|^p(1-r)^{\alpha-p}.$$

$$.r(1+r)^{\alpha}\,drd\theta$$

$$\le\sum_{j=1}^{\infty}\left(1-\left|z_j\right|\right)^{2+\alpha-p}\left(1+\left|z_j\right|\right)^{2+\alpha}.$$

$$.\left|\widetilde{D}_{\left(\hat{f}(n)e^{in\arg\left(z_j\right)}\right)}\left(\sqrt{\left|z_j\right|}\right)\right|^p$$

$$\le C(\alpha+1)\int_0^{2\pi}\int_0^1\left|\widetilde{D}_{\left(\hat{f}(n)e^{in\theta}\right)}\left(\sqrt{r}\right)\right|^p.$$

$$.(1-r)^{\alpha-p}r(1+r)^{\alpha}\,drd\theta.$$

This proves the proposition.

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وصف لفضاءات بلوخ ، فضاءات برغمان الموزونة وفضاء جزئي لا متغير و أسئلة متصلة بها.

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خلاصة

نقوم في هذا البحث بإعطاء وصف جديد لفضاءات بلوخ و فضاءات برغمان الموزونة وذلك باستخدام رموز برزن للمؤثرات القطرية المرتبطة بمعاملات تايلور لدوالها. كما نعطي بدلالة رموز برزن وصفاً للفضاءات الجزئية اللامتغيرة و المتعددة الانزياح لفضاءات برغمان الموزونة . كما نناقش أيضاً بعض الاسئلة الاخرى.