Description of Bloch spaces, weighted Bergman spaces and invariant subspaces, and related questions

Mubariz T. Garayev¹, Mehmet Gürdal²*, Ulaş Yamançlı²

¹Dept. of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail: mgarayev@ksu.edu.sa

²Dept. of Mathematics, Faculty of Arts and Sciences, Süleyman Demirel University, Isparta, Turkey
E-mail: gurdalmehmet@sdu.edu.tr; E-mail: ulasyamanci@sdu.edu.tr

*Corresponding author: gurdalmehmet@sdu.edu.tr

Abstract

Let $\mathbb{D}$ be the unit disc of complex plane $\mathbb{C}$, and $H = Hol(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. Recall that an $f \in Hol(\mathbb{D})$ is said to belong to the Bloch space $B = B(\mathbb{D})$ if $\|f\|_B := \sup_{z \in \mathbb{D}} \left|1 - |z|^2 \right| |f'(z)| < +\infty$. With the norm $\|f\| = |f(0)| + \|f\|_B$, $B$ is Banach space. Let $B_0 = B_0(\mathbb{D})$ be the Bloch space which consists of all $f \in B$ satisfying $\lim_{|z| \to 1^-} \left(1 - |z|^2 \right) |f'(z)| = 0$. Here we give a new description of Bloch spaces and weighted Bergman spaces in terms of Berezin symbols of diagonal operators associated with the Taylor coefficients of their functions. We also give in terms of Berezin symbols a characterization of the multiple shift invariant subspaces of weighted Bergman spaces. Some other questions are also discussed.

Keywords: Berezin symbol; Bloch space; diagonal operator; invariant subspace; weighted Bergman space.

AMS Subject Classification: 47B47, 42A55.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc, and $H = Hol(\mathbb{D})$ the class of functions holomorphic (analytic) in $\mathbb{D}$. We denote by $H^\infty = H^\infty(\mathbb{D})$ the bounded analytic functions space on $\mathbb{D}$. The Bloch space $B$ of $\mathbb{D}$ is defined to be the space of analytic functions $f$ on $\mathbb{D}$ such that

$$\|f\|_B := \sup_{z \in \mathbb{D}} \left(1 - |z|^2 \right) |f'(z)| < +\infty,$$

$\|f\|_B$ is a complete semi-norm on $B$. $B$ can made into a Banach space by introducing the norm

$$\|f\| = |f(0)| + \|f\|_B.$$

The little Bloch space of $\mathbb{D}$, denoted $B_0$, is the closed subspace of $B$ consisting of functions $f$ with

$$\left(1 - |z|^2 \right) f'(z) \to 0 \ (|z| \to 1^-).$$

It is the closure of the set of polynomials in $B$.

Let $C(\overline{\mathbb{D}})$ be the space of continuous functions on the closed unit disc $\overline{\mathbb{D}}$. Denote by $C_0(\mathbb{D})$ the subspace of $C(\overline{\mathbb{D}})$ consisting of functions vanishing on the unit circle $\mathbb{T}$. Clearly, both $C(\overline{\mathbb{D}})$ and $C_0(\mathbb{D})$ are closed subspaces of $L^\infty(\mathbb{D})$.

In this article we give a new characterization of Bloch spaces and some weighted Bergman spaces $A_\alpha^p$ in terms of Berezin symbols of diagonal operators associated with the Taylor coefficients of functions in these spaces. A new characterization of $A_\alpha^p$-interpolating and sampling sequences is given in terms of Berezin symbols. We also characterize in terms of Berezin symbols the multiple shift-invariant subspaces of weighted Bergman spaces. For more detailed and different characterizations of Bloch spaces and weighted Bergman spaces, the reader can refer to Zhu (2007) and Hedenmalm et al. (2000). A characterization of some other function classes is contained in Karaev (2012).

2. Diagonal operators and characterization of Bloch spaces

In the present section, we characterize Bloch spaces in terms of Berezin symbols.

We set

$$Hol(\mathbb{D}) = \{f : f \text{ is analytic on } \mathbb{D}\}.$$

Recall that for a bounded linear operator $A$, acting on the Hardy-Hilbert space $H^2 = H^2(\mathbb{D})$ of analytic functions
$f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ with the sequence of Taylor coefficients

$\hat{f}(n)_{n=0} = \left( \frac{\mathcal{F}^{(n)}(0)}{n!} \right)_{n=0}$ in $\mathcal{L}^2$, its Berezin symbol $\hat{A}$ is defined by

$$\tilde{A}(\lambda) = \left( \mathcal{A} \hat{k}_n(z), \hat{k}_n(z) \right) (\lambda \in D),$$

where $\hat{k}_n(z) = \frac{k_n(z)}{k_n(z)} \frac{(-1)^n}{1-\lambda z}$ is the normalized reproducing kernel of the Hardy space $H^2$. Clearly, $|\tilde{A}(\lambda)| \leq |A|$, and hence $\hat{A}$ is a bounded function on $\mathbb{D}$.

For any bounded sequence $(a_n)_{n=0}$ of complex numbers, let $D(a_n)$ denote the associated diagonal operator on the Hardy space $H^2$ defined by

$$D(a_n)z^k = a_k z^k, \quad k = 0, 1, 2, ... \quad (1)$$

Then, it can be easily shown that (Ash & Karaev, 2012)

$$\tilde{D}(a_n)(\lambda) = \left(1 - |\lambda|^2 \right) \sum_{n=0}^{\infty} a_n |\lambda|^{2n} (\lambda \in D), \quad (2)$$

which means that Berezin symbol $\tilde{D}(a_n)$ of the diagonal operator $D(a_n)$ defined by the formula (1) is a radial function, that is $\tilde{D}(a_n)(\lambda) = \tilde{D}(a_n)(|\lambda|)$ for every $\lambda \in \mathbb{D}$.

The following two results characterize the Bloch spaces functions in terms of their Taylor coefficients, which is one of important and classical questions of the theory of analytic functions (Privalov, 1950; Duren, 2000).

**Theorem 2.1.** Let $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k \in Hol(D)$ be a function with the sequence $(\hat{f}(k))_{k=0}$ of Taylor coefficients $\hat{f}(k) = f^{(k)}(0)/k!$ $(k = 0, 1, 2, ...)$ satisfying $\hat{f}(k + n) = O\left(\frac{k!}{(k + n)!}\right)$ as $k \to +\infty$ for any $n \geq 2$. Then $f \in \mathcal{B}$ if and only if

$$\sup_{0 \leq r < 1, 0 \leq \theta < 2\pi} (1 - r)^{n-1} \left| \mathcal{D}_{\frac{|\lambda|^2}{1-\lambda r}}^{|\lambda|^2} \hat{f}(k+n)|e^{i\theta}| \right|^{\frac{n}{(k+n)!}} < +\infty. \quad (3)$$

Proof. Let $f \in \mathcal{B}$ and $n \geq 1$. Then we have

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \hat{f}(k+n)z^k. \quad (4)$$

By considering that $z = |z|e^{i\theta}(z) = re^{i\theta}$, where $r = |z|$, $\theta = \arg(z)$, and hence $0 \leq r < 1$ and $0 \leq \theta < 2\pi$, we have for $n \geq 2$ from the representation (3) that

$$\left(1 - |z|^2 \right)^n f^{(n)}(z) = (1 - r)^n (1 - r)^{n-1} \left| \mathcal{D}_{\frac{|\lambda|^2}{1-\lambda r}}^{|\lambda|^2} \hat{f}(k+n)|e^{i\theta}| \right|^{\frac{n}{(k+n)!}} \quad (5)$$

By considering the condition of the theorem and formula (2), we obtain from the latter that

$$\left(1 - |z|^2 \right)^n |f^{(n)}(z)| = (1 + r)^n (1 - r)^n \left| \mathcal{D}_{\frac{|\lambda|^2}{1-\lambda r}}^{|\lambda|^2} \hat{f}(k+n)|e^{i\theta}| \right|^{\frac{n}{(k+n)!}} \quad (6)$$

for all $r \in [0,1)$ and $\theta \in [0, 2\pi)$. Since $1 \leq (1 + r)^n < 2^n$, we conclude from the latter that

$$\sup_{0 \leq r < 1, 0 \leq \theta < 2\pi} (1 - r)^{n-1} \left| \mathcal{D}_{\frac{|\lambda|^2}{1-\lambda r}}^{|\lambda|^2} \hat{f}(k+n)|e^{i\theta}| \right|^{\frac{n}{(k+n)!}} < +\infty.$$

For completing the proof, now it remains only to apply the well-known fact that (Zhu, 2007 [Theorem 5.1.5]) if $f$ is analytic in $\mathbb{D}$ and $n \geq 2$, then $f \in \mathcal{B}$ if and only if

$$\sup_{0 \leq r < 1, 0 \leq \theta < 2\pi} (1 - r)^{n-1} \left| \mathcal{D}_{\frac{|\lambda|^2}{1-\lambda r}}^{|\lambda|^2} \hat{f}(k+n)|e^{i\theta}| \right|^{\frac{n}{(k+n)!}} < +\infty,$$

This proves the theorem.

**Theorem 2.2.** Let $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k \in Hol(D)$ be a function and $n$ is a positive integer such that $\hat{f}(k+n) = O\left(\frac{k!}{(k+n)!}\right)$ as $k \to +\infty$. Then $f \in \mathcal{B}$ if and only if

$$\lim_{r \to 1^{-}} (1 - r)^{-n} \left| \mathcal{D}_{\frac{|\lambda|^2}{1-\lambda r}}^{|\lambda|^2} \hat{f}(k+n)|e^{i\theta}| \right|^{\frac{n}{(k+n)!}} = 0 \quad (7)$$

for all $\theta \in [0, 2\pi)$. Proof. The proof is very similar to the proof of Theorem 2.1. Indeed, as in the proof of Theorem 2.1, we obtain that

$$\left(1 - |z|^2 \right)^n f^{(n)}(z) = (1 + r)^n (1 - r)^{-n} \left| \mathcal{D}_{\frac{|\lambda|^2}{1-\lambda r}}^{|\lambda|^2} \hat{f}(k+n)|e^{i\theta}| \right|^{\frac{n}{(k+n)!}} \quad (8)$$
for any \( r \in [0,1) \) and \( \theta \in [0,2\pi) \). Since by condition of the theorem \( \frac{\left| \sum a_n \epsilon^{((\lambda_n-1)\theta)} \right|}{\sqrt{r}} \) is bounded for any fixed \( \theta \in [0,2\pi) \), we conclude that the diagonal operator \( D_{\sum a_n \epsilon^{((\lambda_n-1)\theta)}}(k+1) \) is bounded for any fixed \( \theta \in [0,2\pi) \) and hence its Berezin symbol \( \tilde{D}_{\sum a_n \epsilon^{((\lambda_n-1)\theta)}}(k+1) \) is bounded on \( \mathbb{D} \). Now it remains only to use (5) and the known facts (Hedenmalm et al., 2000 [Proposition 1.13]) if \( n \) is a positive integer and \( f \) is analytic in \( \mathbb{D} \), then \( f \in \mathcal{B} \) and if only if the function \( \left( 1 - |z|^2 \right)^{1/2} f^{(n)}(z) \) is in \( \mathcal{C}(\mathbb{D}) \). Then we have \( \left| \frac{D_{\sum a_n \epsilon^{((\lambda_n-1)\theta)}}(k+1)}{\sqrt{r}} \right| \leq \left( 1 - |z|^2 \right)^{1/2} f^{(n)}(z) \) for all \( \theta \in [0,2\pi) \), which implies that

\[
\sup_{z \in \mathbb{D}} \left( 1 - |z|^2 \right)^{1/2} |f^{(n)}(z)| < +\infty
\]

for all \( \theta \in [0,2\pi) \), and for the compact operator \( D_{\sum a_n \epsilon^{((\lambda_n-1)\theta)}}(k+1) \), vanishes on the boundary.

Recall that a sequence \( (\lambda_n)_n \) of positive integers is called a gap sequence if there exists a constant \( \lambda > 1 \) such that \( \lambda_{n+1} / \lambda_n \geq \lambda \) for all \( n = 1, 2, 3, \ldots \). In this case, we call a power series of the form \( \sum a_n z^{\lambda_n} \) a lacunary series.

Theorem 2.3. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \) be a lacunary power series. Then

(a) \( f \in \mathcal{B} \) if and only if

\[
\sup_{0 < r < 1} \frac{\tilde{D}_{\sum a_n \epsilon^{((\lambda_n-1)\theta)}}(k+1)}{\sqrt{r}} < +\infty.
\]

(b) \( f \in \mathcal{B}_0 \) if and only if

\[
\lim_{r \to 0^+} \frac{\tilde{D}_{\sum a_n \epsilon^{((\lambda_n-1)\theta)}}(k+1)}{\sqrt{r}} = 0
\]

for all \( \theta \in [0,2\pi) \).

Proof. In fact, \( f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{i\lambda_n \theta} \), and therefore

\[
f'(e^{i\theta}) = \sum_{n=0}^{\infty} a_n \lambda_n e^{i\lambda_n \theta} \lambda_n^{-1} = \sum_{n=0}^{\infty} \left( a_n \lambda_n e^{i\lambda_n \theta} \frac{r^{\lambda_n-1}}{r^n} \right) e^{i\lambda_n \theta} r^n = \sum_{n=0}^{\infty} \left( a_n \lambda_n e^{i\lambda_n \theta} \frac{r^{\lambda_n-1}}{r^n} \right) e^{i\lambda_n \theta} r^n.
\]

Then we have

\[
\left( 1 - |z|^2 \right)^{1/2} f'(z) = \left( 1 - r^2 \right) \sum_{n=0}^{\infty} \left( a_n \lambda_n e^{i\lambda_n \theta} \frac{r^{\lambda_n-1}}{r^n} \right) e^{i\lambda_n \theta} r^n
\]

thus

\[
\left( 1 - |z|^2 \right)^{1/2} f'(z) \leq \left( 1 - |z|^2 \right)^{1/2} \sum_{n=0}^{\infty} \left( a_n \lambda_n e^{i\lambda_n \theta} \frac{r^{\lambda_n-1}}{r^n} \right) e^{i\lambda_n \theta} r^n
\]

\[
= \left( 1 + r \right) \left( 1 - r \right) \sum_{n=0}^{\infty} \left( a_n \lambda_n e^{i\lambda_n \theta} \frac{r^{\lambda_n-1}}{r^n} \right) e^{i\lambda_n \theta} r^n
\]

\[
= \left( 1 + r \right) \tilde{D}_{\sum a_n \epsilon^{((\lambda_n-1)\theta)}}(k+1)(\sqrt{r})
\]

\[
\leq \left( 1 - |z|^2 \right)^{1/2} f'(z) \leq 2 \left( D_{\sum a_n \epsilon^{((\lambda_n-1)\theta)}}(k+1) \right)(\sqrt{r}).
\]
with the above norm; when \(0 < p < 1\), the space \(L^p(\mathbb{D}, dA_\alpha)\) is a complete metric space with the metric defined by

\[
d(f,g) = \|f-g\|_{p,\alpha}^p.
\]

We let \(L^\infty(\mathbb{D})\) denote the space of (essentially) bounded functions on \(\mathbb{D}\). For \(f \in L^\infty(\mathbb{D})\) we define

\[
\|f\|_\infty = \text{ess sup}\{f(z) : z \in \mathbb{D}\}.
\]

The space \(L^\infty(\mathbb{D})\) is a Banach space with the above norm. For more informations about weighted Bergman spaces \(\mathcal{A}^p_\alpha\), see, for instance, Hedenmalm et al. (2000).

Our next result characterizes some weighted Bergman spaces in terms of Berezin symbols of diagonal operator corresponding to the Taylor coefficients of functions in these spaces.

**Theorem 3.4.** Suppose \(1 \leq p < +\infty\), \(-1 < \alpha < +\infty\), and that \(n\) is a positive integer. Then an analytic function \(f\) in \(\mathbb{D}\) belongs to \(\mathcal{A}^p_\alpha\) if and only if

\[
2\pi \int_0^1 \int_0^1 (1-r)^{(n-1)p+\alpha} \left| \frac{D_{\|j_{(k,n)}\|}}{\|j_{(k,n)}\|} \right|^p dA_{\alpha}(z) < +\infty.
\]

**Proof.** Let \(f(z) = \sum_{k=0}^{\infty} j(k) z^k\) be an analytic function in \(\mathbb{D}\).

Then by using formula (4), we obtain:

\[
\left| \frac{1-|z|^2}{\alpha+1} \right|^p (1-r)^{(n-1)p+\alpha} \left| \frac{D_{\|j_{(k,n)}\|}}{\|j_{(k,n)}\|} \right|^p dA_{\alpha}(z)
\]

and hence

\[
\int_\mathbb{D} \left| \frac{1-|z|^2}{\alpha+1} \right|^p (1-r)^{(n-1)p+\alpha} \left| \frac{D_{\|j_{(k,n)}\|}}{\|j_{(k,n)}\|} \right|^p dA_{\alpha}(z)
\]

\[
= 2\pi \int_0^1 \int_0^1 (1-r)^{(n-1)p+\alpha} \left| \frac{D_{\|j_{(k,n)}\|}}{\|j_{(k,n)}\|} \right|^p dA_{\alpha}(z)
\]

\[
= \left( \frac{\alpha+1}{\pi} \right) \left( \frac{1-r^2}{\alpha+1} \right)^p \int_0^1 \left| \frac{1-|z|^2}{\alpha+1} \right|^p (1-r)^{(n-1)p+\alpha} \left| \frac{D_{\|j_{(k,n)}\|}}{\|j_{(k,n)}\|} \right|^p dA_{\alpha}(z)
\]

\[
\leq \alpha+1 \int_0^1 \int_0^1 (1-r)^{(n-1)p+\alpha} \left| \frac{D_{\|j_{(k,n)}\|}}{\|j_{(k,n)}\|} \right|^p dA_{\alpha}(z)
\]

On the other hand, since \(1+r \geq 1\), we have

\[
\int_\mathbb{D} \left| \frac{1-|z|^2}{\alpha+1} \right|^p (1-r)^{(n-1)p+\alpha} \left| \frac{D_{\|j_{(k,n)}\|}}{\|j_{(k,n)}\|} \right|^p dA_{\alpha}(z)
\]

\[
\geq \left( \frac{\alpha+1}{\pi} \right) \left( \frac{1-r^2}{\alpha+1} \right)^p \int_0^1 \int_0^1 (1-r)^{(n-1)p+\alpha} \left| \frac{D_{\|j_{(k,n)}\|}}{\|j_{(k,n)}\|} \right|^p dA_{\alpha}(z)
\]

The last two inequalities together with Proposition 1.11 in Hedenmalm et al. (2000), give the proof of the theorem.

The above used method of diagonal operators can also be useful to study the zero sequences of some analytic functions on \(\mathbb{D}\).

**Proposition 3.1.** Let \(f(z) = \sum_{n=0}^{\infty} j(n) z^n\) be an analytic function on \(\mathbb{D}\) with the bounded sequence of Taylor coefficients. If \((\lambda_k)_{k \geq 0}\) is the zero sequence of \(f\), then

\[
\lim_{t \to 1^-} \frac{\bar{D}_{\|j(n)\|}}{\|j(n)\|} \left( \frac{1}{1-t} \right) = 0
\]

for any \(k \geq 0\).

**Proof.** Since \(f(\lambda_k) = 0\), \(k = 0,1,2,\ldots\), it means that \(0 = f(\lambda_k) = \sum_{n=0}^{\infty} j(n) \lambda_k^n\) for any \(k \geq 0\), that is the numerical series \(\sum_{n=0}^{\infty} j(n) \lambda_k^n\) converges to 0 for any \(k \geq 0\). Then it converges to 0 also in Abel sense, which means that \(\sum_{n=0}^{\infty} j(n) \lambda_k^n r^n\) is convergent for all \(r \in (0,1)\) and

\[
\lim_{t \to 1^-} \sum_{n=0}^{\infty} j(n) \lambda_k^n r^n = 0.
\]

Therefore

\[
\lim_{t \to 1^-} \left( \frac{1-t}{1-t} \right) \sum_{n=0}^{\infty} j(n) \lambda_k^n r^n = 0
\]

for any \(k \geq 0\), and hence it follows from (2) that

\[
\lim_{t \to 1^-} \bar{D}_{\|j(n)\|} \left( \frac{1}{1-t} \right) = 0
\]

for any \(k \geq 0\). This proves the proposition.

**Remark 3.1.** (a) Since \(\lambda_k = 0\) for any \(k \geq 0\), we have that \(\lim_{t \to 1^-} \bar{D}_{\|j(n)\|} \left( \frac{1}{1-t} \right) = 0\). So, condition (7) means some growth condition for the boundary values of the Berezin symbols \(\bar{D}_{\|j(n)\|}\) of compact operators \(D_{\|j(n)\|}\), \(k = 0,1,2,\ldots\).
If \( \{\hat{f}(n)\}_{n=0}^{\infty} \) in addition satisfies the Tauberian type condition that \( \hat{f}(n) = O\left(\frac{1}{n}\right), n \to \infty \), then the necessary condition (7) is also sufficient in order to a sequence \( (\lambda_k)_{k=0}^{\infty} \) be a zero sequence of \( f \).

(c) Proposition 3.1, apparently, may also be useful in investigation of the zeros of the Riemann \( \xi \)-function.

### 4. Description of multiple shift invariant subspaces

In this short section, we use the diagonal operators technique in description of invariant subspaces of multiple shift operator \( M_z f = z^nf \) on the weighted Bergman spaces.

Theorem 4.5. Let \( E \subset A^p_\alpha \), \( 1 \leq p < \infty, -1 < \alpha < +\infty \), be a nonzero proper subspace, and \( n \geq 1 \) be any integer. Then \( M_z E \subset E \) if and only if for every \( f \in E \) there exists a function \( g = g_f \in E \) such that

\[
\tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha}(z) = (1 - |z|)^{a}\tilde{g}(z), \quad z \in \mathbb{D}.
\]

Proof. For any \( f \in E \) we have:

\[
z^n f(z) = z^n \sum_{k=0}^{\infty} \hat{f}(k)z^k = \sum_{k=0}^{\infty} \hat{f}(k-n)z^k
\]

\[
= \sum_{k=n}^{\infty} \hat{f}(k-n)e^{i\alpha \arg(z)}|z|^k
\]

\[
= \frac{(1 - |z|)\sum_{k=0}^{n} \hat{f}(k-n)e^{i\alpha \arg(z)}|z|^k}{1 - |z|}
\]

(we put \( \hat{f}(-n) = \hat{f}(\ldots(n-1)) = \ldots = \hat{f}(-1) := 0 \)). So, it follows from formula (2) that

\[
z^n f(z) = \frac{\tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha}(z)}{1 - |z|}, \quad z \in \mathbb{D}.
\]

Now formula (9) implies that \( z^n f \in E \) if and only if

\[
\tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha}(z) \in E,
\]

which means that

\[
\tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha}(z) = (1 - |z|)^{a}\tilde{g}(z)
\]

for some \( g = g_f \in E \). Thus, \( z^n E \subset E \) if and only if for any \( f \in E \) there is a function \( g = g_f \in E \) satisfying (8), as desired.

Recall that the class \( \ell^\infty_\alpha = \ell^\infty(\mathbb{D}) \) consists of analytic functions \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \) on \( \mathbb{D} \) with \( \{\hat{f}(n)\}_{n=0}^{\infty} \in \ell^\infty \). Recall also that the range of the Berezin symbol \( \tilde{A} \) is called the Berezin set of the operator \( A \), and will be denoted as \( \text{Ber}(A) \), i.e., \( \text{Ber}(A) = \{\tilde{A}(\lambda) : \lambda \in \mathbb{D}\} \). Clearly, \( \text{Ber}(A) \subset W(A) \), where \( W(A) \) is the numerical range of operator \( A \) defined as \( W(A) = \{\langle Af, f \rangle : \|f\|_{H^p} = 1\} \). For any subspace \( E \subset \ell^\infty_\alpha \), we set

\[
B_E = \bigcup_{f \in E} \text{Ber}(\tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha})
\]

and

\[
B_E = \bigcup_{f \in E} \text{Ber}(\tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha})
\]

The following result belongs to Ash & Karaev (2012); below \( S \) denotes the usual shift operator defined by \( Sf = zf \).

Lemma 4.1. If \( f \in \ell^\infty_\alpha \), then \( \text{Range}(1 - |z|, f) = \bigcup_{\theta \in [0, 2\pi)} \text{Ber}(\tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha}) \).

Our next result describes \( z \)-invariant (shift-invariant) subspaces in terms of the sets \( B_E \) and \( B_{zE} \).

Theorem 4.6. Let \( E \subset A^p_\alpha \) be a nontrivial (closed) subspace (i.e., \( 0 \notin E \neq \mathbb{D} \)). Then \( SE \subset E \) if and only if \( B_E \subseteq B_{zE} \).

Proof. It follows from Theorem 4.5 for \( n = 1 \) that \( SE \subset E \) if and only if for any \( f \in E \) there exits a function \( g = g_f \in E \) such that

\[
\tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha}(z) = (1 - |z|)^{a}\tilde{g}(z) \quad (\forall z \in \mathbb{D}).
\]

By Lemma 4.1, we have then that

\[
\text{Range}(1 - |z|, \tilde{g}(z)) = \bigcup_{\theta \in [0, 2\pi)} \text{Ber}(\tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha})
\]

So, for any \( z \in \mathbb{D} \) there exists \( w = w_z \in \mathbb{D} \) such that

\[
(1 - |z|)^{a}\tilde{g}(z) = \tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha}(w)
\]

On the other hand, since \( g \in zE \), we have \( \hat{g}(0) = 0 \) and \( \hat{g}(z) = \sum_{k=0}^{\infty} \hat{g}(k)z^k \) for any \( k \geq 1 \). Then, by considering this and formulas (10) and (11), we obtain

\[
\tilde{D}_{\{\hat{f}(n)\}_n}^{a,\alpha}(z) = \tilde{D}_{\{\hat{g}(n)\}_n}^{a,\alpha}(w).
\]

Or, equivalently, \( SE \subset E \) if and only if \( \forall f \in E, \exists g \in zE \) such that
for all \( \theta \in [0, 2\pi) \). This means that \( SE \subset E \) if and only if
\[
\bigcup_{\theta \in [0, 2\pi)} Ber \left( D_{(j-k)e^{i\theta}} \right) \subseteq Ber \left( D_{(\delta_j e^{i\theta})} \right),
\]
or \( B_E \subseteq B_{SE} \). This proves the theorem.

5. Remarks on interpolation and sampling in \( A^p \)

In this brief section, we show how the technique of diagonal operators of the previous sections can be adapted in study of interpolating and sampling sequences for the weighted Bergman spaces \( A^p \).

Recall that a sequence \( \{z_j\}_j \) of distinct points in \( \mathbb{D} \) is called an \( A^p \)-interpolation sequence (or a sequence of interpolation for \( A^p \)) if for every sequence \( \{w_j\}_j \) of complex numbers satisfying the condition
\[
\sum_j \left( 1 - |z_j|^2 \right)^{2+\alpha} |w_j|^p < +\infty,
\]
there exists a function \( f \in A^p \) such that \( f(z_j) = w_j \) for all \( j \) (Hedenmalm et al., 2000). Then we have:
\[
w_j = f(z_j) = \sum_{n=0}^{\infty} \tilde{f}(n) z^n_j = \sum_{n=0}^{\infty} \tilde{f}(n) e^{i \arg(z_j)} |z_j|^n
\]
\[
eq \frac{\left( 1 - |z_j|^2 \right)^{2+\alpha} |w_j|^p}{1 - |z_j|^2}.
\]

Thus
\[
\tilde{D}(\tilde{f}(n) e^{i \arg(z_j)} |z_j|^n)
\]
\[
= \frac{\left( 1 - |z_j|^2 \right)^{2+\alpha} |w_j|^p}{1 - |z_j|^2}.
\]

and hence
\[
(1 - |z_j|^2) w_j = \tilde{D}(\tilde{f}(n) e^{i \arg(z_j)} |z_j|^n)
\]
for all \( j \). Therefore
\[
\left( 1 - |z_j|^2 \right)^{2+\alpha} |w_j|^p
\]
\[
= (1 + |z_j|) \left( 1 - |z_j|^2 \right)^{1+\alpha} |w_j|^p.
\]

Since \( 1 \leq 1 + z_j < 2 \), the latter implies the proof of the following.

Proposition 5.2. A sequence \( \Gamma = \{z_j\}_j \) of distinct points in \( \mathbb{D} \) is an \( A^p \)-interpolation sequence if and only if
\[
\sum_j \left( 1 - |z_j|^2 \right)^{2+\alpha} |f(z_j)|^p \tilde{D}(\tilde{f}(n) e^{i \arg(z_j)} |z_j|^n) < +\infty
\]
for every function \( f \) in \( A^p \).

Recall that a sequence \( \Gamma = \{z_j\}_j \) of (not necessarily distinct) points in \( \mathbb{D} \) is called an \( A^p \)-sampling sequence (or a sequence of sampling for \( A^p \)) if there exists a positive constant \( C \) such that
\[
C^{-1} \int_\mathbb{D} |f(z)|^p dA(z)
\]
\[
\leq \sum_{j=1}^{\infty} \left( 1 - |z_j|^2 \right)^{2+\alpha} |f(z_j)|^p
\]
\[
\leq C \int_\mathbb{D} |f(z)|^p dA(z)
\]
for all \( f \in A^p \) (Hedenmalm et al., 2000).

Proposition 5.3. A sequence \( \Gamma = \{z_j\}_j \) of (not necessarily distinct) points in \( \mathbb{D} \) is an \( A^p \)-sampling sequence if and only if there exists a positive constant \( C \) such that
\[
C^{-1} \int_\mathbb{D} |f(z)|^p dA(z)
\]
\[
\leq \sum_{j=1}^{\infty} \left( 1 - |z_j|^2 \right)^{2+\alpha} |f(z_j)|^p
\]
\[
\leq C \int_\mathbb{D} |f(z)|^p dA(z)
\]
for all \( f \in A^p \).

Proof. Let \( f \in A^p \) be arbitrary. Then by using our previous calculus, we have
\[
f(z) = \tilde{D}(\tilde{f}(n) e^{i \arg(z_j)} |z_j|^n)
\]
\[
= \frac{\left( 1 + |z_j|^2 \right)^{2+\alpha} |w_j|^p}{1 - |z_j|^2}.
\]

From (13), we obtain
\[
\int_\mathbb{D} |f(z)|^p dA(z)
\]
\[
= (\alpha + 1) \int_\mathbb{D} \left( 1 - |z|^2 \right)^{\alpha} dA(z)
\]
\[
= (\alpha + 1) \int_\mathbb{D} \tilde{D}(\tilde{f}(n) e^{i \arg(z_j)} |z_j|^n) \left( 1 - |z|^2 \right)^{\alpha} drd\theta
\]
\[
= (\alpha + 1) \int_0^{2\pi} \int_0^r \tilde{D}(\tilde{f}(n) e^{i \arg(z_j)} |z_j|^n) \left( 1 - r^2 \right)^{\alpha} r drd\theta
\]
\[
= (\alpha + 1) \int_0^{2\pi} \tilde{D}(\tilde{f}(n) e^{i \arg(z_j)} |z_j|^n) \left( 1 - r^2 \right)^{\alpha} r drd\theta.
\]
and also
\[ \sum_{j=1}^{\infty} \left( 1 - |z_j|^2 \right)^{2+\alpha} |f(z_j)|^p \]
\[ = \sum_{j=1}^{\infty} \left( 1 - |z_j|^2 \right)^{2+\alpha} \left( \frac{\bar{D}_{\gamma_j}^{\infty, m}(\sqrt{|z_j|})}{1 - |z_j|} \right)^p. \]

Since again \( 1 \leq (1+r)^p < 2^\alpha \) and \( 1 \leq (1+r)^{2+\alpha} < 2^{2+\alpha} \), by considering all these we have from the definition of \( A^p_\alpha \)-sampling sequence that a sequence \( \Gamma = \{z_j\}_j \) is a \( A^p_\alpha \)-sampling sequence if and only if there exists a positive constant \( C \) such that
\[ C^{-1}(\alpha + 1) \int_0^{2\pi} \int_0^1 \bar{D}_{\gamma_j}^{\infty, m}(\sqrt{r})^p (1-r)^{\alpha-p} \cdot \]
\[ r(1+r)^{\alpha} dr d\theta \leq \sum_{j=1}^{\infty} \left( 1 - |z_j|^2 \right)^{2+\alpha-p} \left( 1 + |z_j|^2 \right)^{2+\alpha} \]
\[ \left( \bar{D}_{\gamma_j}^{\infty, m}(\sqrt{|z_j|}) \right)^p \]
\[ \leq C(\alpha + 1) \int_0^{2\pi} \int_0^1 \bar{D}_{\gamma_j}^{\infty, m}(\sqrt{r})^p \cdot \]
\[ (1-r)^{\alpha-p} r(1+r)^{\alpha} dr d\theta. \]

This proves the proposition.

Acknowledgements

This work is supported by King Saud University, Deanship of Scientific Research, College of Science Research Center. Also, the second author is supported by TUBA through Young Scientist Award Program (TUBA-GBIP/2015).

References


Submitted: 27/02/2015
Revised: 05/05/2015
Accepted: 21/05/2015
وصف لفضاءات بلوخ، فضاءات برغمان الموحدة وفضاء جزئي لا متغير وآسية متصلة بها.

المؤلفين:
1. مبارك قاراييف، مهتم جاردل، إليس يامانس
2. معهد الرياضيات والميكانيكا الأكاديمية الوطنية للعلوم في أذربيجان، فاجياد B شارع 9، باكو 370143، أذربيجان
2. قسم الرياضيات، كلية العلوم - جامعة الملك سعود - الرياض 1451 - المملكة العربية السعودية
2. جامعة سليمان ديربل - قسم الرياضيات - الرياض 12260 - المملكة العربية السعودية
3. البريد الإلكتروني: ulasyamanci@sdu.edu.tr, gurdalmehmet@sdu.edu.tr

خلاصة

تقوم في هذا البحث بإعطاء وصف جديد لفضاءات بلوخ وفضاءات برغمان الموحدة وذلك باستخدام رموز برزن للمؤثرات القطرية المرتبطة بعمليات تايلور لدوالها. كما تعطي بدالة رموز برزن وصفًا للفضاءات الجزئية الامتدائية والمتميزة لفضاءات برغمان الموحدة، كما نناقش أيضاً بعض الأسئلة الأخرى.