Kuwait J.Sci., Vol.48 (2), April.2021, pp(1-11)

On the gamma spectrum of multiplication gamma acts

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Abstract

In this paper, we introduce the concept of topological gamma acts as a generalization of Zariski topology. Some topological properties of this topology are studied. Various algebraic properties of topological gamma acts have been discussed. We clarify the interplay between this topological space's properties and the algebraic properties of the gamma acts under consideration. Also, the relation between this topological space and (multiplication, cyclic) gamma act was discussed. We also study some separation axioms and the compactness of this topological space.

Keywords: Gamma spectrum; multiplication gamma acts; prime gamma subact; radical gamma subact; topological gamma acts.

1. Introduction

In 1981, Sen, M. (Sen, 1981) introduced the concept of gamma semigroups as a generalization of semigroups as follows: let S and Γ be nonempty sets, S is said to be a gamma semigroup (Γ-semigroup for short) if there is a mapping: S × Γ × S → S written (s_1, α, s_2) by $s_1 \alpha s_2$ that satisfies the condition $s_1\alpha(s_2\beta s_3)=(s_1\alpha s_2)\beta s_3$ for all s_1, s_2, $s_3 \in$ S and $\alpha, \beta \in \Gamma$. Let S be a Γ-semigroup. An element $s \in$S is called the left (right) identity of S if $s\alpha t=t$ ($t\alpha s = t$) for all $t \in S$ and $\alpha \in \Gamma$. An element s in S is called identity if it is both a left and right identity of S. A Γ-semigroup with identity is called a Γ-monoid. The identity of a Γ-semigroup (if exists) is denoted by 1. A Γ-semigroup S is called commutative if $s\alpha t=t\alpha s$ for all $s, t\in$ S and $\alpha \in \Gamma$. A nonempty subset T of Γ-semigroup S is called a Γ-subsemigroup of S if $s\alpha t \in$ T, for all $s, t \in$ T and $\alpha \in \Gamma$. A nonempty subset A of semigroup S is called left (right) Γ-ideal if SΓA $\subseteq$ A (AΓS$\subseteq$ A) where SΓA=$\{s\alpha a: s \in S, \alpha \in \Gamma$ and $a\in$A$\}$. The word Γ-ideal is used for a two-sided Γ-ideal. The union of any family of Γ-ideals of Γ-semigroup S is a Γ-ideal of S (Sen, 1981). An element $s \in$ S is said to be α-idempotent if there exists α∈Γ such that $s\alpha s=s$. A Γ-semigroup S is called idempotent if all elements of S are α-idempotent. For any subsets A and B of S, then, AΓB=$\{a\alpha b: a \in$ A, $b \in$ B, and $\alpha\in\Gamma\}$. A Γ-ideal B of a Γ-semigroup S is called globally idempotent (gl-idempotent for short) if BΓB=B (Anjaneyulu *et al.*, 2012). A Γ-ideal P of S is said to be prime provided that for any two Γ-ideals A, B of S with AΓB ⊆ P, either A ⊆ P or B ⊆ P (Anjaneyulu *et al.*, 2011). A Γ-ideal B of a Γ-semigroup S is called maximal if it is proper and is not properly contained in any proper Γ-ideal of S (Anjaneyulu *et al.*, 2012). In 2016, Abbas M. and Faris A. (Abbas & Faris, 2016) introduced gamma's concept over gamma semigroup as follows: let S be a Γ-semigroup. A nonempty set M is called left gamma act over S (denoted by S_Γ-act) if there is a mapping: S × Γ × M → M defined by $(s, \alpha, m) \mapsto s\alpha m$, satisfying $(s_1\alpha s_2)\beta m$ $=s_1\alpha(s_2\beta m)$ for all s_1, $s_2 \in$ S, $\alpha, \beta \in \Gamma$ and $m \in$ M. In the same way, we can define right gamma acts. From now on, "S_Γ-act" means left S_Γ-act. A nonempty subset N of a left SΓ-act M is called gamma subact (denoted by SΓ-subact) if, for all $s \in S$, $\alpha \in \Gamma$ and $n \in$ N implies that $s\alpha n \in$ N. An element θ ∈ M is called a zero of M if $s\alpha\theta=\theta$, and if S is a Γ-semigroup with zero then, $0\alpha m=\theta$ for all $m \in$ M and $\alpha\in\Gamma$. Let N be a SΓ-subact of SΓ-act M. Then, $[N:M]=\{s\in S|s\alpha m\in N$ for all $\alpha\in\Gamma$ and $m\in M\}$. Clearly, [N: M] is a Γ-ideal of S. Given a family of SΓ-subacts $\{N_i\}_{i\in I}$ of SΓ-act M. Then, $\bigcup_{i\in I} N_i$ is SΓ-subact of M, and if $\bigcap_{i\in I} N_i$ is nonempty, then, $\bigcap_{i\in I} N_i$ is SΓ-subact of M. Let M and N be two SΓ-acts. A mapping f:M → N is called SΓ-homomorphism if $f(s\alpha m)=s\alpha f(m)$ for every $s \in S$, $\alpha \in\Gamma$ and $m\in$ M. If f is surjective, then, f is SΓ-epimorphism. Let f:M→N be SΓ-homomophism. Then, the kernel f is defined as $\ker(f) = \{(m_1, m_2)\in M\times M \,|\, f(m_1) = f(m_2)\}$ (Kamal, 2016). An equivalence relation ρ on SΓ-act M is called a congruence if for all $(m_1, m_2)\in \rho$ implies that $(s\alpha m_1,\ s\alpha m_2)\in\rho$ for all $s \in S$, $\alpha \in \Gamma$. Also, the quotient gamma act of the congruence ρ on M is denoted by M/ρ define by $M/\rho=\{m\rho|m\in M$ and $m\rho$ the equivalent class containing $m\}$. If N is SΓ-subact of M, then, N/ρ_N is a SΓ-subact of M/ρ where $\rho_N = \rho \cap (N \times N)$. If H is a nonempty subset of SΓ-act M, then, $\ell_S(H) = \{(s, t)\in S\times S|s\alpha h = t\alpha h$ for all $\alpha \in \Gamma$ and $h \in H\}$. It is known that $\ell_S(H)$ is a congruence on SΓ-act S (Kamal, 2016). Recently, Abbas M. and Jubeir S. (Abbas & Jubeir, 2020) introduced the concept of multiplication gamma acts. An SΓ-act M is said to be a multiplication if every SΓ-subact N of M is of the form N=AΓM for some Γ-ideal A of S. An SΓ-act M is multiplication if and only if N=[N:M]ΓM for all SΓ-subact N of M. Let M be a SΓ-act and $s_1, s_2 \in$ S. Then, M is called faithful if the equality $s_1\alpha m=s_2\alpha m$ implies that $s_1=s_2$ for every $m \in$ M and α ∈ Γ. Let S be a commutative Γ-monoid and M be a faithful SΓ-act. Then, M is a multiplication if and only if $\bigcap_{i\in I} (A_i\Gamma M)=(\bigcap_{i\in I} A_i)\Gamma M$ for any nonempty collection of Γ-ideals A_i, $i \in I$ of S, and for all SΓ-subact N of M and Γ-ideal A of S such that N ⊂ AΓM there exists an Γ-ideal B with B ⊂ A and N ⊆ BΓM. Let A be a Γ-ideal of Γ-monoid S, and M be a SΓ-act. If M is faithful multiplication, then, A=[AΓM:M] (Abbas & Jubeir, 2020). Let N_1, N_2 be SΓ-subacts of multiplication SΓ-act M. If N_1=AΓM and N_2=BΓM for some Γ-ideals A and B of S, then the product of N_1 and $N_2$is denoted by $N_1 * N_2$ is defined by $N_1 * N_2 = (A\Gamma B)\Gamma M$. Clearly, $N_1 * N_2$ is an SΓ-subact of M. Let N be a SΓ-subact of multiplication SΓ-act M. Then, N is called gamma nilpotent(Γ-nilpotent for short) if $N^k=\theta$ for some positive integer k, where N^k means the product of N, k times. (Abbas & Adnan, 2020).

For SΓ-act M, the set of all prime SΓ-subacts of M is called the gamma spectrum of M and denoted by $Spec_\Gamma(M)$. We remark that $Spec_\Gamma(\theta)=\emptyset$ and that $Spec_\Gamma(M)$ may be empty; for example, the zero SΓ-act has no prime SΓ-subact. Throughout this paper, we assume that $Spec_\Gamma(M)$ is nonempty. This article aims to study topological gamma acts for which the gamma spectrum is a topology in which the varieties $V_\Gamma(N)=\{P \in Spec_\Gamma(M): N \subseteq P\}$ are closed sets for any SΓ-subacts N of the SΓ-act M. Note that our definition is a generalization of the Zariski topology on the spectrum of prime ideals of a ring. Thus, we extend the well-known results of Zariski topology on $Spec(R)$ to $Spec_\Gamma(M)$ and investigate the basic properties of this topology. The concepts of semiprime and extraordinary SΓ-subacts are introduced to identify some cases when the gamma spectrum of a gamma act forms a topology. Also, using the concept of multiplication gamma acts to investigate various algebraic properties of such topology. We prove that $Spec_\Gamma(M)$ is a T_0-space and it is compact if M is finitely generated multiplication gamma act. The relationship between $Spec_\Gamma(M)$ and $Spec_\Gamma(S)$ was investigated.

2. Preliminaries

In this section we introduce the concept of prime gamma subacts and basic related concepts which are needed in our work.

2.1. Definition.

A proper SΓ-subact N of M is prime if for any $m\in$ M and $s \in$ S, the set inclusion

$s\Gamma S\Gamma m \subseteq$ N implies either $m \in$ N or $s \in$ [N:M].

2.2. Example.

Let S={∅, {a},{b},{c},{a,b},{b,c}, {a,c}, {a,b,c}}, Γ={∅,{a},{a ,b ,c}} and M=S. Then, M is a S_Γ-act under the mapping: S × Γ × M → M defined by (A, B, C) ↦ A∩B∩C. It can be easily verified that the S_Γ-subact {{a, b},{a}, ∅} is a prime S_Γ-subact of M.

2.3. Proposition.

Let S be a commutative Γ-semigroup and N be a proper S_Γ-subact of S_Γ-act M. The following statements are equivalent:

i. For every S_Γ-subact K of M, if N ⊂ K, then, [N:M]=[N:K].

ii. N is prime.

Proof:(i)⟹(ii) Let $s\in$S and $m\in$M, such that $s\Gamma S\Gamma m\subseteq$ N and $m \notin$ N. It is clear that N ⊂ N ∪ $S\Gamma m$. Since $s\Gamma(N\cup S\Gamma m) \subseteq s\Gamma N \cup s\Gamma(S\Gamma m)\subseteq$N. By statement (i), $s\in$[N: N∪ $S\Gamma m$] = [N:M].

(ii)⟹(i) Let K be a S_Γ-subact of M such that N⊂K. Clearly [N:M]⊆[N:K]. Now, suppose that $s\in$[N:K]. Then, $s\Gamma S\Gamma K \subseteq s\Gamma K \subseteq$ N. Since N⊂ K,then, there exists $k\in$ K\N such that $s\Gamma S\Gamma k \subseteq$N. By statement(ii), $s\in$[N:M]. Hence, [N:M]=[N:K].

2.4. Proposition.

Let S be a commutative Γ-monoid, and M be a multiplication S_Γ-act. Then, for any S_Γ-subact N of M, the following conditions are equivalent:

i. N is prime S_Γ-subact of M.

ii. [N:M] is prime Γ-ideal of Γ-semigroup S.

iii. There exists a prime Γ-ideal P of S, which is maximal with the property PΓM=N.

Proof: (i)⟹(ii) Let A, B be Γ-ideals of S. Consider the inclusion AΓB ⊆ [N:M].Then, (AΓB)ΓM ⊆ N. Now, assume A ⊈ [N:M] then there is $a\notin$[N:M] such that a$\alpha x \notin$ N, for some $x\in$M and $\alpha\in\Gamma$. Let b∈ B, then, bΓ$(a\alpha x)$=$(b\Gamma a)\alpha x$=$(a\Gamma b)\alpha x\subseteq$N. Since N is a prime S_Γ-subact of M and $a\alpha x \notin$ N, then, bΓM ⊆ N. Thus, [N:M] is prime.

(ii)⟹(iii) Consider the family of Γ-ideals T={P:N=PΓM and P is an Γ-ideal of S}. Since M is multiplication S_Γ-act, then, T is a nonempty partial order set by the usual inclusion relation. Let $\{P_i\}_{i\in I} \subseteq$ T be a chain. Then, $\bigcup_{i\in I} P_i \in$ T is an upper bound of $\{P_i\}_{i\in I}$. Zorn's Lemma implies that T has a maximal element such as P(say). Now, let A and B be two Γ-ideals such that AΓB⊆ P, then, (AΓB)ΓM⊆ PΓM ⊆ N and hence we obtain AΓB⊆[N:M], but by the assumption that [N:M] is prime we conclude that either A⊆ [N:M]⊆P or B⊆[N:M]⊆P. Thus P is prime Γ-ideal of S.

(iii)⟹(i) Let P be a prime Γ-ideal of S, which is maximal with the property PΓM ⊆ N. Clearly, P=[N:M]. Let $x\in$S and $m\in$M such that $x\Gamma S\Gamma m \subseteq$ N. Since M is a multiplication, then there exists a Γ-ideal A of S, such that $S\Gamma m$=AΓM, and hence $x\Gamma$(AΓM)= $x\Gamma S\Gamma m$ ⊆ N. Thus $x\Gamma$A ⊆ [N:M]. Since [N:M]=P and P is prime Γ-ideal of S, then, $x\in$[N:M] or A⊆[N:M] i.e $x \in$ [N:M] or $m \in S\Gamma m$=AΓM ⊆ N. Thus, N is prime S_Γ-subact of M.

2.5. Proposition.

Let N be a proper $S_\Gamma$subac of S_Γ-act M. Then, N is a prime in M if and only if N/ρ_N is prime S_Γ-subact of S_Γ-act M/ρ.

Proof:(⟹)Let $s\in$S and $m\rho\in M/\rho$ where $m\in$ M satisfy $s\Gamma S\Gamma(m\rho)\subseteq N/\rho_N$. Thus $(s\alpha t\beta m)\rho$ $=s\alpha t\beta(m\rho)\in N/\rho_N$ for all $t\in$S and $\alpha,\beta\in\Gamma$. This implies that $s\alpha t\beta m \subseteq$N. By hypothesis ,$s\Gamma$M⊆N or $m\subseteq$N. Thus, $s\Gamma(M/\rho)$=$s\Gamma M/\rho \subseteq N/\rho_N$ or $m\rho \in N/\rho_N$.

(⟸) Let $s \in$ S and $m \in$ M such that $s\Gamma S\Gamma m\subseteq$N. Then, $s\alpha t\beta m \in$ N for all $t\in$S and $\alpha, \beta \in \Gamma$. So, $s\alpha t\beta(m\rho_N)$=$(s\alpha t\beta m)\rho_N$ $\in N/\rho_N$. It follows that by assumption $s\in$ $[M/\rho\text{: } N/\rho_N]$ or $m\rho_N\in N/\rho_N$. Thus, $s\Gamma M/\rho$ $\subseteq N/\rho_N$ or $m \in$N. Hence, $s\gamma m_\circ \in$ N for all $m_\circ \in$ M and $\gamma\in\Gamma$. Thus, $s\Gamma$M ⊆ N or $m \in$ N.

2.6. Proposition.

Let $\{N_i, i\in I\}$ be a non-empty collection of S_Γ-subacts of S_Γ-act M.

If N_i is a prime SΓ-subacts for each $i\in I$, then, $\bigcup_{i\in I} N_i$ is a prime SΓ-subact of M. Proof: Let s∈ S and $m \in M$ satisfy $s\Gamma S\Gamma m \subseteq \bigcup_{i\in I} N_i$. Then, $s\Gamma S\Gamma m \subseteq N_j$ for some $j \in I$. By hypothesis, either $s\in[N_j:M]$ or $m\in N_j$. So, $s \in [\bigcup_{i\in I} N_i:M]$ or $m \in \bigcup_{i\in I} N_i$ and hence $\bigcup_{i\in I} N_i$ is a prime SΓ-subact.

In the following, we characterize prime SΓ- subact in multiplication SΓ-act by product SΓ- subacts.

2.7. Theorem.

Let S be a Γ-monoid and P be a proper SΓ-subact of a multiplication SΓ-act M. Then P is prime if and only if $N_1 * N_2 \subseteq P$ then either $N_1 \subseteq P$ or $N_2 \subseteq P$ for each SΓ-subacts N_1, N_2 of M.

Proof: (⟹) Let P be a prime and $N_1 * N_2 \subseteq P$, but neither $N_1 \not\subseteq P$ nor $N_2 \not\subseteq P$ for some SΓ-subacts N_1, N_2 of M. Since M is a multiplication, then $N_1=A\Gamma M$ and $N_2=B\Gamma M$ for some Γ-ideals A, B of S. So there is a∈A ,b∈ B and $\alpha,\beta\in\Gamma$ such that $a\alpha m\in N_1\backslash P$ and $b\beta m'\in N_2\backslash P$. Since $(A\Gamma B)\Gamma M\subseteq P$. Thus $a\alpha(b\beta m')\in P$, and since P is prime then either $a\in[P:M]$ that is, $a\alpha m\in P$ or $b\beta m'\in P$, which is a contradiction.

(⟸) Let $s \in S$ and $x \in M$ such that $s\Gamma S\Gamma x \subseteq P$. Suppose that $x \notin P$. Let $m\in M$. Since M is multiplication, then $S\Gamma m= A\Gamma M$ and $S\Gamma(s\alpha x)=B\Gamma M$ for some Γ-ideals A, B of S. Now, $S\Gamma(s\alpha x)*S\Gamma m=(B\Gamma A)\Gamma M \subseteq B\Gamma M = S\Gamma(s\alpha x)\subseteq s\Gamma S\Gamma x\subseteq P$. By hypothesis, we have $S\Gamma(s\alpha x) \subseteq P$ or $S\Gamma m \subseteq P$, but $x \notin P$ so $s\Gamma M \subseteq P$. Hence, P is prime.

Let M be a multiplication SΓ-act and m, $m' \in M$. Then $S\Gamma m=A\Gamma M$ and $S\Gamma m'=B\Gamma M$ for some Γ-ideals A, B of S. So $m * m'$ means the product of $S\Gamma m$ and $S\Gamma m'$, which is equal to $(A\Gamma B)\Gamma M$. As consequence of Theorem(2.7), we give the following Corollary:

2.8. Corollary.

Let S be Γ-monoid and P be a proper SΓ- subact of a multiplication SΓ-act M. Then P is prime if and only if $m * m' \in P$ then either $m \in P$ or $m' \in P$ for each m, $m' \in M$.

2.9. Definition.

Let N be a SΓ-subact of SΓ-act M. Then the radical of N is the intersection of all prime SΓ-subacts of M containing N and denoted by $rad_M(N)$. If N is not contained in any prime SΓ-subact of M, then $rad_M(N)=M$.

2.10. Proposition.

Let S be a commutative Γ-monoid and N be a proper SΓ-subact of a faithful multiplication SΓ-act M. Then $rad_M(N)=\sqrt{[N:M]}\,\Gamma M$.

Proof: Let $\mathcal{F} = \{P: P \text{ is prime } \Gamma\text{- ideal of S such that } [N:M] \subseteq P\}$. Let $B=\sqrt{[N:M]}$ then $B=\bigcap_{P\in\mathcal{F}} P$ and hence, $B\Gamma M=\bigcap_{P\in\mathcal{F}}(P\Gamma M)$. Let $P \in \mathcal{F}$. If $M=P\Gamma M$ then $rad_M(N) \subseteq P\Gamma M$. If $M\neq P\Gamma M$ then $N=[N:M]\Gamma M \subseteq P\Gamma M$ by Proposition (2.4), PΓM is prime SΓ-subact of M. Thus $rad_M(N)\subseteq P\Gamma M$. Therefore, $rad_M(N) \subseteq B\Gamma M$. Conversely, let K be a prime SΓ-subact of M containing N. Then by Proposition (2.4), there exists a prime Γ-ideal P of S, such that $K=P\Gamma M$ and Since $[N:M]\Gamma M=N\subseteq K=P\Gamma M$ then $[N:M]\subseteq P$, and hence $\sqrt{[N:M]}\subseteq P$. So, $\sqrt{[N:M]}\Gamma M \subseteq P\Gamma M=K$. Thus, $\sqrt{[N:M]}\Gamma M \subseteq rad_M(N)$.

Now, we give the concept of completely globally idempotent as follows:

2.11. Definition.

A Γ-semigroup S, is called completely globally idempotent if every Γ-ideal of S, is gl-idempotent.

2.12. Example.

Let S=Γ= {i, 0, -i}. Then S is Γ-semigroup under the multiplication over complex numbers. Here, $A_1=\{0\}$ and $A_2=S$ are the only Γ-ideals of S. It's clear that A_1 and A_2 are gl-idempotent.

3. Topological gamma acts

In this section, we introduce the concept of a topological gamma act and its basic properties discussed. In what follows, S will denote a Γ-semigroup with zero, and all SΓ-acts contain the zero element.

3.1. Definition.

Let M be a SΓ-act. The gamma spectrum (Γ-spectrum for short) of M is the collection of prime SΓ-subacts of M and denoted by $Spec_\Gamma$(M). When S is an SΓ-act, then $Spec_\Gamma$(S) is the set of all prime Γ-ideals of S.

3.2. Definition.

Let N be a SΓ-subact of SΓ-act M. We define V_Γ(N) to be the set of all prime SΓ-subacts of M containing N, i.e $V_\Gamma(N)=\{P \in Spec_\Gamma(M): N \subseteq P\}$. Note that, V_Γ(M) is empty set and $V_\Gamma(\theta)$ is $Spec_\Gamma$(M).

It's easy to see that for SΓ-subacts N_1 and N_2 of M we have:

i. If $N_1 \subseteq N_2$, then $V_\Gamma(N_2) \subseteq V_\Gamma(N_1)$.

ii. If $V_\Gamma(N)=\{N\}$, then N is a prime. The converse is true if N is the unique prime SΓ-subact of M.

iii. $V_\Gamma(N_1) \cup V_\Gamma(N_2) \subseteq V_\Gamma(N_1 \cap N_2)$.

3.3. Proposition.

Let S be a Γ-monoid and M be a SΓ-act. Then for the SΓ-subacts N, N_1 and N_2 of M, the following conditions hold:

i. If M is a multiplication then, $V_\Gamma(N_1)\cup V_\Gamma(N_2)=V_\Gamma(N_1 * N_2)=V_\Gamma(N_1 \cap N_2)$.

ii. $V_\Gamma(rad_\Gamma(N)) = V_\Gamma(N)$.

iii. If $V_\Gamma(N_1) \subseteq V_\Gamma(N_2)$, then $N_2 \subseteq rad_\Gamma(N_1)$.

iv. $V_\Gamma(N_1)=V_\Gamma(N_2)$ if and only if $rad_\Gamma(N_1) = rad_\Gamma(N_2)$.

v. $N_1=N_2$ for any SΓ-subacts N_1 , N_2 of M whenever $V_\Gamma(N_1)=V_\Gamma(N_2)$ is equivalent to every proper SΓ-subact N is the intersection of primes.

vi. If $\{N_i: i \in I\}$ is a nonempty collection of SΓ-subacts of M, then $\bigcap_{i\in I} V_\Gamma(N_i)=V_\Gamma(\bigcup_{i\in I} N_i)$.

Proof:(i) Let $P \in V_\Gamma(N_1)\cup V_\Gamma(N_2)$. Then $P\in V_\Gamma(N_1)$ or $P \in V_\Gamma(N_2)$ and hence $N_1 \subseteq P$ or $N_2\subseteq P$. Since $N_1 * N_2 \subseteq N_1$ and $N_1 * N_2 \subseteq N_2$ thus $P \in V_\Gamma(N_1 * N_2)$. Conversely, let $P' \in V_\Gamma(N_1 * N_2)$ then $N_1 * N_2 \subseteq P'$. By Theorem (2.7), $N_1 \subseteq P'$ or $N_2 \subseteq P'$ that is, $P' \in V_\Gamma(N_1) \cup V_\Gamma(N_2)$. For the other part let $Q \in V_\Gamma(N_1 * N_2)$ then $N_1 * N_2 \subseteq Q$. Thus $N_1 \subseteq Q$ or $N_2 \subseteq Q$ which implies that $Q \in V_\Gamma(N_1 \cap N_2)$. The other direction is clear.

(ii) For $P \in V_\Gamma(N)$ we have $N \subseteq P$ and hence $rad_\Gamma(N)\subseteq P$. So $P\in V_\Gamma(rad_\Gamma(N))$. Conversely $P' \in V_\Gamma(rad_\Gamma(N))$ then $rad_\Gamma(N) \subseteq P'$. Since $N \subseteq rad_\Gamma(N)$. Thus, $P' \in V_\Gamma(N)$. Therefore, $V_\Gamma(rad_\Gamma(N))= V_\Gamma(N)$.

(iii) By hypothesis, $N_2 \subseteq P$ for every $P \in V_\Gamma(N_1)$. Thus $N_2 \subseteq \bigcap_{P\in V_\Gamma(N_1)}P$ and hence $N_2 \subseteq rad_\Gamma(N_1)$.

(iv) Clearly, $rad_\Gamma(N_1)=\bigcap_{N_1\subseteq P}P=\bigcap_{P\in V_\Gamma(N_1)}P =\bigcap_{P\in V_\Gamma(N_2)}P=rad_\Gamma(N_2)$. Conversely, by (ii) $V_\Gamma(N_1)=V_\Gamma(rad_\Gamma(N_1)=V_\Gamma(rad_\Gamma(N_2)=V_\Gamma(N_2)$.

(v)(⟹)Suppose N_1 is a proper SΓ-subact of M. If $V_\Gamma(N_1)=\emptyset$, then $V_\Gamma(N_1)=V_\Gamma(M)$. By hypothesis N_1=M, a contradiction. Thus $V_\Gamma(N_1) \neq\emptyset$, we obtain $N_2=\bigcap_{P\in V_\Gamma(N_1)}P=rad_\Gamma(N_1)$. Then, $V_\Gamma(N_2)=V_\Gamma(rad_\Gamma(N_1)=V_\Gamma(N_1)$, since $N_1=N_2$. Hence, N_1 is an intersection of prime SΓ-subacts.

(⟸) Assume that $V_\Gamma(N_1)=V_\Gamma(N_2)$. It's clear by hypothesis every SΓ-subact N of M is an intersection of prime SΓ-subacts if and only if $N=\bigcap_{P\in V_\Gamma(N)}P$. Now, $N_1=\bigcap_{P\in V_\Gamma(N_1)}P$ and $N_2=\bigcap_{P\in V_\Gamma(N_2)}P$. It follows that $N_1= N_2$.

(vi) Let $P \in \bigcap_{i\in I} V_\Gamma(N_i)$. Then $P \in V_\Gamma(N_i)$ and hence $N_i\subseteq P$ for all $i \in I$. Thus, $\bigcup_{i\in I} N_i \subseteq P$. So, P∈ $V_\Gamma(\bigcup_{i\in I} N_i)$. Conversely, let Q∈$V_\Gamma(\bigcup_{i\in I} N_i)$. Then, $\bigcup_{i\in I} N_i \subseteq$ Q. We conclude that $N_i \subseteq$ Q and hence Q∈ $V_\Gamma(N_i)$ for every i∈I. Thus, Q∈ $\bigcap_{i\in I} V_\Gamma(N_i)$. Hence, $\bigcap_{i\in I} V_\Gamma(N_i) = V_\Gamma(\bigcup_{i\in I} N_i)$.

Now, we introduce the definition of topological gamma act as follows:

3.4. Definition.

Let M be a S$_\Gamma$-act, and τ(M)={V_Γ(N): N is S$_\Gamma$-subact of M}. If τ(M) is closed under finite unions, then the family τ(M) satisfies the axioms for the closed subsets of a topological space. Thus,
τ(M) is a topology on $Spec_\Gamma$(M) called the gamma act topology.

3.5. Examples.

1. Let S=M={w, x, y, z} and Γ any nonempty set. Then, M is an S$_\Gamma$-act under the multiplication mapping: S×Γ×M ⟶ M defined by aαb=ab, which given in the following table:

.	w	x	y	z
w	w	w	w	w
x	w	w	w	x
y	w	w	w	w
z	w	w	y	z

Here {w},{w, x},{w, y}and{w, x,y}are the S$_\Gamma$- subacts of M. But, {w, x, y} is the only prime S$_\Gamma$-subact of M. So, $Spec_\Gamma$(M)={{w, x, y}}, V_Γ({w})=V_Γ({w, x})=V_Γ({w, y})=$Spec_\Gamma$(M)and V_Γ(M)=∅. So, τ(M)={∅, $Spec_\Gamma$(M)}.In this case τ(M) is the indiscrete topological S$_\Gamma$-act.

2. Let S = $\mathbb{Z}_6$, Γ={$\bar{1}, \bar{3}$}, and M=S. Clearly, M is an S$_\Gamma$-act under multiplication mod 6. The S$_\Gamma$-subacts of M are N_1={$\bar{0}$}, N_2={$\bar{0}$, $\bar{2}$, $\bar{4}$}, N_3={$\bar{0}$, $\bar{3}$}, and N_4={$\bar{0}$, $\bar{2}$, $\bar{3}$, $\bar{4}$}. It's clear that N_2 ,N_3 , N_4 are the only prime S$_\Gamma$-subacts of M.
Thus, $Spec_\Gamma$(M)={N_2, N_3, N_4},
$V_\Gamma(N_1)$=$Spec_\Gamma$(M), V_Γ(M)=∅, $V_\Gamma(N_2)$={$N_2 N_4$}, $V_\Gamma(N_3)$={N_3,N_4} and $V_\Gamma(N_4)$={N_4}. Hence, τ(M)={∅,$Spec_\Gamma$(M), {N_2, N_4}, {N_3, N_4},{N_4}}.

3. Let S=M={a, b, c, d, e, f} and Γ be a nonempty set. Then M is an S$_\Gamma$-act under the mapping: S×Γ×M ⟶ M, which defined by:

$$x\,\alpha\,y = \begin{cases} a & if \quad x = f, y = a \\ b & if \quad x = f, y = b \\ c & if \quad x = f, y = c \\ f & if \; x = f = y \\ e & if \; x \in M, y = b \; or \; y = e \\ d & otherwise. \end{cases}$$

Here, the S$_\Gamma$-subacts of M are: N_1={a, d, e, f}, N_2={b, d, e, f}, N_3={c, d, e, f}, N_4= {b, c, d, e, f} , N_5={a, c, d, e, f}, N_6={a, b, d, e, f} and N_7=M. Then it can be easily verified that, $V_\Gamma(N_1)$={ N_1, N_6, N_5}, $V_\Gamma(N_2)$={N_2, N_6, N_4},$V_\Gamma(N_3)$={ N_3, N_4, N_5},$V_\Gamma(N_4)$={N_4}, $V_\Gamma(N_5)$={N_5}and $V_\Gamma(N_6)$= {N_6}.
But $V_\Gamma(N_5) \cup V_\Gamma(N_6)$={ N_6, N_5}≠$V_\Gamma(N_i)$ for all i=1,2,3,4 ,5,6 and $V_\Gamma(N_5) \cup V_\Gamma(N_6) \neq V_\Gamma(N_5 \cap N_6)$.
Now, we identify some cases for which τ(M) is a topological gamma act. Before this, we need the following definitions.

3.6. Definition.

An S$_\Gamma$-subact N of S$_\Gamma$- act M is called semiprime if N is an intersection of prime S$_\Gamma$-subacts of M.

3.7. Definition.

A prime S$_\Gamma$- subact N of S$_\Gamma$- act M is said to be extraordinary if whenever K and L are semiprime S$_\Gamma$- subacts of M with K∩L ⊆ N then K ⊆ N or L ⊆ N.

3.8. Example.

Let S = Γ =$\mathbb{Z}$ and M = 6$\mathbb{Z}$. Then M is a S$_\Gamma$-act under the usual multiplication of integer numbers.

It is clear that $Spec_\Gamma$(M)={$(6p)\Gamma S$: p∈ℙ where ℙ is the set of prime numbers}. Thus, any semiprime SΓ-subact N of M has a form N=$\bigcap_{i\in I}(6p_i)$ΓS, for some p_i ∈ ℙ. Also, the prime SΓ-subact P=(12)ΓS of M is extraordinary. Let K and L be semiprime SΓ-subacts of M, such that K=$\bigcap(6p)$ΓS and L=$\bigcap(6q)$ΓS, where the intersection runs among some p, q in ℙ. Thus, K=$(6m)$ΓS and L=$(6n)$ΓS where m, n ∈ ℤ. Hence, K∩L =$(6nm)$ΓS⊆P=6(2)ΓS.So, we have 2 divides nm then 2 divides n or 2 divides m. This, implies that K ⊆ P or L ⊆ P.

3.9. Theorem.

For a SΓ-act M. The following conditions are equivalent:

i. M is a Top SΓ-act.

ii. Every prime SΓ-subact of M is extraordinary

iii. V_Γ(N)∪V_Γ(K)=V_Γ(N∩K) for all semiprime SΓ-subacts N and K of M.

Proof:(i)⟹(ii) Let K be a prime SΓ-subact of M and N, L be semiprime SΓ-subacts of M such that N∩L⊆K. By assumption, there is an SΓ-subact T of M such that V_Γ(N)∪V_Γ(L)=V_Γ(T). Since N is semiprime, then there exists a collection of prime SΓ-subacts K_i , (i∈ I) such that, N=$\bigcap_{i\in I} K_i$. So, for all i ∈I, K_i∈ V_Γ(N) ⊆V_Γ(T), and hence T⊆ K_i for all $i \in I$. Thus, T⊆$\bigcap_{i\in I} K_i$=N . Similarly T ⊆ L. So T⊆N∩L. Now, V_Γ(N)∪V_Γ(L)⊆V_Γ(N∩L)⊆V_Γ(T)=V_Γ(N)∪ V_Γ(L). We conclude that V_Γ(N)∪V_Γ(L) =V_Γ(N∩L). But, K∈V_Γ(N∩L). Thus, K ∈ V_Γ(N) or K∈ V_Γ(L). Hence, N ⊆K or N⊆L. Therefore, K is extraordinary.

(ii)⟹(iii) Suppose that N and K semiprime SΓ-subacts of M. It's clear that $V_\Gamma(N)$ ∪ $V_\Gamma(K)$⊆$V_\Gamma(N\cap K)$. Let L ∈ $V_\Gamma(N\cap K)$. Then, $N\cap K$⊆ L. Since L is prime then by (ii), N⊆L or K⊆ L, $i.e$ L ∈ $V_\Gamma(N)$ or L∈ $V_\Gamma(K)$. This shows that $V_\Gamma(N\cap K)$⊆ V_Γ(N) ∪ V_Γ(K). Hence, V_Γ(N) ∪ V_Γ(K) = V_Γ(N∩K).

(iii) ⟹(i) Let K_1 and K_2 be any SΓ-subacts of M. If $V_\Gamma(K_1)$ is empty, then $V_\Gamma(K_1)$∪ $V_\Gamma(K_2)$= $V_\Gamma(K_2)$. Assume that $V_\Gamma(K_1)$ and $V_\Gamma(K_2)$ are both nonempty. Then, $V_\Gamma(K_1)$ ∪ $V_\Gamma(K_2)$=$V_\Gamma(rad_\Gamma(K_1))$∪ $V_\Gamma(rad_\Gamma(K_2))$ =$V_\Gamma(rad_\Gamma(K_1)\cap rad_\Gamma(K_2))$. Thus, M is a Top SΓ-act.

If { A_i, 1≤i≤n} is any nonempty family of Γ-ideals of Γ-semigroup S, then , $A_1\Gamma A_2\Gamma$ ... ΓA_n⊆ $A_1\cap A_2\cap ... \cap A_n$.

3.10. Corollary.

Let M be a SΓ-act with the property that for every prime SΓ-subact N of M, [K:M]⊆[N:M] implies that K⊆N for each semiprime SΓ-subact K of M. Then M is a Top SΓ-act.

Proof: Let K_1, K_2 be a semiprime SΓ-subacts of M with $K_1\cap K_2$ ⊆ N. It follows that [K_1 :M]∩[K_2 :M]=[K_1 ∩K_2 :M]⊆[N:M], since [N:M] is a prime Γ-ideal of S, then either [K_1 :M]⊆[N:M] or [K_2 :M]⊆[N:M]. Now, by hypothesis, we have K_1 ⊆ N or K_2 ⊆ N, that is, N is extraordinary. Hence, M is a Top SΓ-act by Theorem (3.9).

3.11. Corollary.

Any homomorphic image of a Top SΓ-act is a Top SΓ-act.

Proof: Consider the SΓ-epimorphism π:M⟶ M/ρ, where ρ a congruence on M. Let N/ρ_N be a prime SΓ-subact of SΓ-act M /ρ, where N is a prime SΓ-subact of M (by Proposition (2.5)). Consequently, any semiprime SΓ-subact of M/ρ is of the form K/ρ_K for which K is semiprime SΓ-subact of M. Let K_1/ρ_{K_1} and K_2/ρ_{K_2}be semiprime SΓ-subac of M/ρ such that $(K_1/\rho_{K_1}\cap K_2/\rho_{K_2})$⊆ N/$\rho_N$. Then , $\pi^{-1}(K_1/\rho_{K_1}\cap K_2/\rho_{K_2}) \subseteq \pi^{-1}(N/\rho_N)$. So $\pi^{-1}(K_1/\rho_{K_1})\cap\pi^{-1}(K_2/\rho_{K_2}) \subseteq \pi^{-1}(N/\rho_N)$. Thus, $K_1\cap K_2$ ⊆N. It follows that K_1⊆N or K_2⊆N and hence, $\pi(K_1)\subseteq\pi(N)$ or

$\pi(K_2) \subseteq \pi(N)$. Thus, $K_1/\rho_{K_1} \subseteq N/\rho_N$ or $K_2/\rho_{K_2} \subseteq N/\rho_N$. Hence, N/ρ_N is extraordinary. So, by Theorem (3.9), M/ρ is Top S_Γ-act.

Now, let P be a prime Γ-ideal of Γ-semigroup S and A_1, A_2 be a semiprime Γ-ideals of S with $A_1 \cap A_2 \subseteq P$. Since $A_1\Gamma A_2 \subseteq A_1 \cap A_2 \subseteq P$, then $A_1 \subseteq P$ or $A_2 \subseteq P$ and hence P is extraordinary. Thus, by Theorem (3.9) any Γ-semigroup S is a Top S_Γ-act .

Recall, if M and N are S_Γ-acts, f:M → N S_Γ-homomorphism, then M/ker(f)≅Im(f). (Kamal, 2016)

Now, we study the relation between cyclic and topological gamma acts. For this reason, we give the following Proposition.

3.12. Proposition.

Let M be a S_Γ-act. If M is cyclic, then $M \cong S/\ell_S(m)$ for some $m \in$ M. Proof: Let M be a cyclic S_Γ-act. Then, there exists $m \in$ M such that $M = S\Gamma m$. Define, $f: S \longrightarrow M$ by $f(s) = s\alpha m$ for every s∈ S and $\alpha \in \Gamma$. Now, let s, s'∈ S and $\beta \in \Gamma$, thus $f(s\beta s') = (s\beta s')\alpha m = s\beta(s'\alpha m) = s\beta f(s')$ and hence f is a S_Γ-homomorphism. Also, let $m' \in$M, then $m' = t\gamma m = f(t)$. Hence, M≅ S/ker(f) .Since ker(f)=$\{(s,t) \in S \times S | f(s) = f(t)\} = \{(s,t) \in S \times S | s\alpha m = t\alpha m\} = \ell_S(m)$. Thus, $M \cong S/\ell_S(m)$.

3.13. Corollary.

Any cyclic S_Γ-act is a Top S_Γ-act.

Proof: It's clear by Corollary (3.11) and Proposition (3.12).

But the converse of Corollary (3.13) may not be true, as we can see in the first part of example (3.5).

3.14. Corollary.

Let S be a Γ-semigroup and T be a Γ-sub-semigroup of S . If M is a Top T_Γ-act, then M is a Top S_Γ-act.

Proof: Let K be a prime S_Γ-subact of M. Then K is a proper T_Γ-subact of M. If $t, s \in$T, $m \in$ M and $\alpha, \beta \in \Gamma$ satisfy $t\alpha s\beta m \in$K then $m \in$ K or $t\Gamma M \subseteq$ K. Thus K is a prime T_Γ-subact of M. Let L_1 and L_2 be semiprime S_Γ-subacts of M with $L_1 \cap L_2 \subseteq$ K. By the same way above, we have L_1 and L_2 are semiprime T_Γ-subact of M. Since M is a Top T_Γ-act then, $L_1 \subseteq$ K or $L_2 \subseteq$ K. Thus, K is extraordinary. Hence, by Theorem (3.9), M is a Top S_Γ-act.

3.15. Proposition.

Let A be a Γ-ideal of Γ- monoid S and N be a S_Γ-subact of S_Γ-act M. Then $V_\Gamma(N) \cup V_\Gamma(A\Gamma M) = V_\Gamma(A\Gamma N) = V_\Gamma(N \cap A\Gamma M)$.

Proof: It's clear that, $V_\Gamma(N) \cup V_\Gamma(A\Gamma M) \subseteq V_\Gamma(N \cap A\Gamma M) \subseteq V_\Gamma(A\Gamma N)$. Let $P \in V_\Gamma(A\Gamma N)$. Then $A\Gamma N \subseteq P$. This implies that $A\Gamma 1\Gamma N \subseteq P$ and hence $A\Gamma S\Gamma N \subseteq P$. Since P is a prime, then $N \subseteq P$ or $A\Gamma M \subseteq P$. So, $P \in V_\Gamma(N)$ or $P \in V_\Gamma(A\Gamma M)$. Thus, $P \in V_\Gamma(N) \cup V_\Gamma(A\Gamma M)$. Therefore, $V_\Gamma(A\Gamma M) \subseteq V_\Gamma(N) \cup V_\Gamma(A\Gamma M)$.

3.16. Corollary.

Let A and B be a Γ-ideals of Γ-monoid S and M be a S_Γ-act. Then $V_\Gamma(A\Gamma M) \cup V_\Gamma(B\Gamma M) = V_\Gamma[(A\Gamma B)\Gamma M]$.

Now, if $\{A_i\}_{i \in I}$ is a collection of Γ-ideals of S, then it's easy to show that, $\bigcap_{i \in I} V_\Gamma(A_i\Gamma M) = V_\Gamma(\bigcup_{i \in I} A_i\Gamma M)$. Thus, by using this fact with Corollary (3.16), we get the subset $\tau(A\Gamma M) = \{V_\Gamma(A\Gamma M)$: where A is Γ-ideal of S$\}$ of τ(M) is a topological space, and if M is a Top S_Γ-act, then $\tau(A\Gamma M)$ is a subspace of τ(M). In particular, if M is a multiplication then M is a Top S_Γ-act. Thus, we have the following result:

3.17. Proposition.

Ever multiplication S_Γ-act is a Top S_Γ-act.

3.18. Corollary.

If S is completely globally idempotent Γ-semigroup then any Γ-ideal of S is a Top S_Γ-act.

Proof: Let A and B be Γ-ideals of S, such that B ⊆ A. Then B = BΓB ⊆ BΓA ⊆ B. Hence, B=BΓA. Thus, A is a multiplication. By Proposition (3.17), A is Top S_Γ-act .

We denote the complement of V_Γ(N) in τ(M) for any S_Γ-subact N of M by D_Γ(N), i.e. $D_\Gamma(N) = Spec_\Gamma(M) \setminus V_\Gamma(N)$. Note that $D_\Gamma(m) = D_\Gamma(S\Gamma m)$ for every $m \in M$.

3.19. Proposition.

If M is a multiplication S_Γ-act, then the following conditions hold:

i. $D_\Gamma(m) \cap D_\Gamma(n) = D_\Gamma(m * n)$ for any $m, n \in M$.

ii. Let N⊂M with every proper S_Γ-subact of M is the intersection of primes. If D_Γ(N) =∅, then N is Γ-nilpotent .

Proof:(i) Let $m, n \in M$. Then $S\Gamma m$=AΓM and $S\Gamma n$=BΓM for some Γ-ideals A and B of S. By Corollary (3.16), $D_\Gamma(m) \cap D_\Gamma(n) = D_\Gamma(A\Gamma M) \cap D_\Gamma(B\Gamma M) = [Spec_\Gamma(M) \setminus V_\Gamma(A\Gamma M)] \cap [Spec_\Gamma(M) \setminus V_\Gamma(B\Gamma M)] = Spec_\Gamma(M) \setminus [V_\Gamma(A\Gamma M) \cup V_\Gamma(B\Gamma M)] = Spec_\Gamma(M) \setminus V_\Gamma((A\Gamma B)\Gamma M) = Spec_\Gamma(M) \setminus V_\Gamma(m * n) = D_\Gamma(m * n)$.

(ii) Let $D_\Gamma(N)$=∅. Then, $Spec_\Gamma(M) \setminus V_\Gamma(N)$= ∅ thus $Spec_\Gamma(M) = V_\Gamma(N)$ and hence $V_\Gamma(N) = V_\Gamma(\theta)$. By Proposition (3.3)(v), N=θ. Therefore, N is a Γ-nilpotent.

3.20. Proposition.

Let M be a S_Γ-act. Then the sets $\{D_\Gamma(m_i) : i \in I\}$ forms a base of the gamma act topology on M.

Proof: Any nonempty open set in the gamma act topology contains D_Γ(K) for some S_Γ-subact K of M. Now, any such K= $\bigcup_{i\in I}\{m_i\}$, $m_i \in K$. Then, $D_\Gamma(K) = D_\Gamma(\bigcup_{i\in I} m_i) = Spec_\Gamma(M) \setminus V_\Gamma(\bigcup_{i\in I} m_i) = Spec_\Gamma(M) \setminus \bigcap_{i\in I} V_\Gamma(m_i) = \bigcup_{i\in I} D_\Gamma(m_i)$.

3.21. Definition

(Erdogan, 2003) Let $\mathcal{A} = \{\_A_i : i \in I\}$ be a collection of sets. Then $\mathcal{A}$ is said to have the finite intersection property if for every finite collection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$, we have that $\bigcap_{i=1}^{n} A_i \neq \emptyset$.

3.22. Theorem

(Erdogan, 2003) A topologicalspace X is compact if and only if for every collection of closed sets $\mathcal{A}$ of X, with $\mathcal{A}$ has the finite intersection property then, $\bigcap_{A\in\mathcal{A}} A \neq \emptyset$.

3.23. Theorem.

If M is finitely generated multiplication S_Γ-act, then $Spec_\Gamma(M)$ is compact.

Proof: Let $\{V_\Gamma(N_i) : i \in I\}$ be any collation of closed subsets of $Spec_\Gamma(M)$ where N_i is a S_Γ-subact of M for each $i \in I$ such that $\bigcap_{i\in I} V_\Gamma(N_i) = \emptyset$. Thus, by Theorem (3.3)(vi), $\bigcap_{i\in I} V_\Gamma(N_i) = V_\Gamma(\bigcup_{i\in I} N_i) = V_\Gamma(\bigcup_{i\in I}[N_i : M]\Gamma M)$ and hence $V_\Gamma(\bigcup_{i\in I}[N_i : M]\Gamma M) = \emptyset$. Now, suppose that $M \neq \bigcup_{i\in I}[N_i : M]\Gamma M$ then $V_\Gamma(M) \neq V_\Gamma(\bigcup_{i\in I}[N_i : M]\Gamma M)$. This implies that $\emptyset \neq V_\Gamma(\bigcup_{i\in I}[N_i : M]\Gamma M)$, a contradiction. Thus, $\bigcup_{i\in I}[N_i : M]\Gamma M = M$. Since M is finitely generated, there exists a finite subset I' of I such that $M = \bigcup_{i\in I'}[N_i : M]\Gamma M$. Therefore , $\bigcap_{i\in I'} V_\Gamma(N_i) = V_\Gamma(\bigcup_{i\in I'} N_i) = V_\Gamma(\bigcup_{i\in I'}[N_i : M]\Gamma M) = \emptyset$. Which contradicts the finite intersection property. By Theorem (3.22), $Spec_\Gamma(M)$ is compact.

Now, we study some of the separation axioms and the density of topological gamma acts.

3.24. Definition.

Let M be a S_Γ-act and X be a nonempty subset of $Spec_\Gamma(M)$. Then the Jacobson radical of X is the intersection of all prime S_Γ-subacts of M which belong to X and denoted by J_Γ(X). We denote the closure of a subset X of $Spec_\Gamma(M)$ by $\overline{X}$.

3.25. Theorem.

Let M be a Top S_Γ-act. Then, $\overline{X} = V_\Gamma((J_\Gamma(X))$.

Proof: Let V_Γ(N) be a closed set containing X, and P be a prime S_Γ-subact in X. Then N ⊆P, and hence N⊆J_Γ(X) .Thus, V_Γ(J_Γ(X))⊆V_Γ(N). Since X ⊆ V_Γ (J_Γ(X)), then V_Γ((J_Γ(X)) is the smallest closed subset of $Spec_\Gamma$(M) containing X. So, $\overline{X}$=V_Γ((J_Γ(X)).

Recall that a topological space is a T_0-space if and only if the closures of distinct points are distinct. A subset A of a topological space X is called dense in X if $\overline{X}$=A. (Erdogan 2003 & Oner 2020)

3.26. Corollary.

If (θ) ∈ X, then X is dense in $Spec_\Gamma$(M).
Proof: By Theorem (3.25), $\overline{X}$ = V_Γ ((J_Γ(X)) = $V_\Gamma(\theta)$=$Spec_\Gamma$(M). Hence, X is dense.

3.27. Corollary.

$Spec_\Gamma$(M) is a T_0-space for every Top S_Γ-act M.
Proof: Let N_1 and N_2 be two distinct points of $Spec_\Gamma$(M). Then, $\overline{\{N_1\}}$=$V_\Gamma(N_1)$≠$V_\Gamma(N_2)$=$\overline{\{N_2\}}$. (By Theorem (3.25)). We deduce that, $Spec_\Gamma$ (M) is a T_0-space.

A topological space X is a T_1-space if and only if all points of X are closed in X (i.e., given any x in X, the singleton set {x} is a closed set. (Erdoğan 2003&Öner 2020).

3.28. Theorem.

Let M be a S_Γ-act. Then $Spec_\Gamma$ (M) is T_1-space if and only if each prime S_Γ-subact in $Spec_\Gamma$(M) is maximal.
Proof:(⟸)Let{P}⊆ $Spec_\Gamma$(M).Then, $\overline{\{P\}}$=V_Γ ((J_Γ({P}))=V_Γ(P). Since {P} is maximal. Thus, $\overline{\{P\}}$ = V_Γ(P) ={P}.
(⟹) Let P be a prime S_Γ-subact of M. By hypothesis {P} is a closed subset of $Spec_\Gamma$(M). Thus $V_\Gamma(P)$=V_Γ ((J_Γ(P))=$\overline{\{P\}}$ = P. Hence, P is maximal in $Spec_\Gamma$(M).
Recall, if N is a prime S_Γ-subact of S_Γ-act M, then [N:M] is a prime Γ-ideal of S. This idea motivates us to introduce and study the following mapping that gives a relationship between $Spec_\Gamma$(M) and $Spec_\Gamma$(S). Before this we need the following Lemma.

3.29. Lemma.

Let S be a commutative Γ-monoid and M be a S_Γ-act. Then [SΓP:M] =SΓ[P:M] for all S_Γ-subact P of M.
Proof: Clear.

3.30. Definition.

Let S be a Γ-monoid and M be a S_Γ-act. Define a mapping ψ: $Spec_\Gamma$(M) ⟶ $Spec_\Gamma$(S), by P ⟼ [P:M] for all P ∈ $Spec_\Gamma$(M).

Clearly, by Lemma (3.30) ψ is well-defined and S_Γ-homomorphism. The next Proposition present some properties of the mapping ψ .

3.31. Proposition.

Let S be a Γ–monoid, and M be a multiplication S_Γ-act, then
i. If M is faithful, then mapping ψ is surjective.
ii. The mapping ψ is injective.
Proof:(i) Let P ∈ $Spec_\Gamma$(S). By Proposition (2.4), PΓM is a prime S_Γ-subact. Now, ψ(PΓM)=[PΓM:M]=P.
(ii) Let N_1, N_2∈$Spec_\Gamma$(M) with $\psi(N_1)$=$\psi(N_2)$. Then, [N_1:M]=[N_2:M] and hence [N_1:M]ΓM = [N_2:M]ΓM. So, N_1=N_2 .
Thus, the mapping ψ plays an important role in studying algebraic properties of the S_Γ-act M when we have a related topology. For an example, if M is a faithful multiplication S_Γ-act, then $Spec_\Gamma$(S) and $Spec_\Gamma$(M) are homeomorphic, and hence we can transfer some of known topological properties of $Spec_\Gamma$(M) to $Spec_\Gamma$(S) .

ACKNOWLEDGEMENTS

The authors would like to thank Al-Mustaqbal University College and Al-Zahraa University for Women for giving them the opportunity time and scientific support for completing this research.

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Submitted	**:** 13/07/2020
Revised	**:** 10/09/2020
Accepted	**:** 13/09/2020
DOI	**:** 10.48129/kjs.v48i2.10147