Symmetric and generating functions of generalized (*p*,*q*)-numbers

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Abstract

In this paper, we will firstly define a new generalization of numbers (p, q) and then derive the appropriate Binet's formula and generating functions concerning (*p*,*q*)-Fibonacci numbers, (*p*,*q*)-Lucas numbers, (*p*,*q*)-Pell numbers, (*p*,*q*)-Pell Lucas numbers, (*p*,*q*)-Jacobsthal numbers and (*p*,*q*)-Jacobsthal Lucas numbers. Also, some useful generating functions are provided for the products of (*p*,*q*)-numbers with bivariate complex Fibonacci and Lucas polynomials.

Keywords: Bivariate complex fibonacci polynomials; generalized (*p*,*q*)-numbers; generating functions; (*p*,*q*)-Jacobsthal numbers; symmetric functions.

1. Introduction and preliminaries

Recently, Gulec & Taskara (2012) defined and studied the (*p*,*q*)-Pell and (*p*,*q*)-Pell Lucas numbers. Accordingly, they showed some interesting properties of these numbers. For their part, Suvarnamani & Tatong (2015) investigated the (*p*,*q*)-Fibonacci numbers of the form $\left\{F_{p,q,n}\right\}_{n\in\mathbb{N}}$ where they studied and analyzed some results using the well-known Binet's formula. Furthermore, Suvarnamani (2016) derived some useful properties of the (*p*,*q*)-Lucas numbers and provided also in another paper (Suvarnamani, 2017) some novel identities for the (*p*,*q*)-Fibonacci numbers using the matrix methods.

On the other hand, Uygun (2015) defined and studied both the (*p*,*q*)-Jacobsthal and (*p*,*q*)-Jacobsthal Lucas numbers $\left\{J_{p,q,n}\right\}_{n\in\mathbb{N}}$ and $\left\{j_{p,q,n}\right\}_{n\in\mathbb{N}}$ by giving some important relationships between these two numbers.

Note that Asci & Gurel (2012) have investigated the bivariate complex Fibonacci and bivariate complex Lucas polynomials $F_n(x,y)$ and $L_n(x,y)$ using the following recurrence relations:

$F_{n+1}(x,y)=ixF_n(x,y)+yF_{n-1}(x,y)$, for $n\geq 1$,

where $F_0(x,y)=0$ and $F_1(x,y)=1$, and

$L_{n+1}(x,y)=ixL_n(x,y)+yL_{n-1}(x,y)$, for $n\geq 1$,

where $L_0(x,y)=2$ and $L_1(x,y)=ix$.

Consequently, the Binet's formulas for the previous recurrence formulas are given by:

$$F_n(x,y)=\frac{\alpha^n(x,y)-\beta^n(x,y)}{\alpha(x,y)-\beta(x,y)},$$

and

$$L_n(x,y)=\alpha^n(x,y)+\beta^n(x,y),$$

where $\alpha(x,y)=\frac{ix+\sqrt{-x^2+4y}}{2}$ and $\beta(x,y)=\frac{ix-\sqrt{-x^2+4y}}{2}$ represent the roots of the characteristic equation $z^2-ixz-y=0.$ We note that $\alpha(x,y)+\beta(x,y)=ix$ and $\alpha(x,y)\beta(x,y)=-y$, (see (Asci & Gurel, 2012)).

Theorem 1.1 *The bivariate complex Fibonacci polynomials can be formulated explicitly as follows*

$$F_n(x,y)=\sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-j-1}{j}(ix)^{n-2j-1}y^j.$$

Proof. The generating function for bivariate complex Fibonacci polynomials is given by

$$g(z)=\sum_{n=0}^{\infty}F_n(x,y)z^n=\frac{z}{1-ixz-yz^2},$$

Thus,

$$\sum_{n=0}^{\infty}F_n(x,y)z^n=\frac{z}{1-(ixz+yz^2)}$$

$$=z\sum_{n=0}^{\infty}(ixz+yz^2)^n$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(ix)^{n-j}y^jz^{n+j+1}.$$

Writing n instead of $n+j+1$, we obtain

$$\sum_{n=0}^{\infty}F_n(x,y)z^n=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-j-1}{j}(ix)^{n-2j-1}y^j\right)z^n,$$

and finally we deduce

$$F_n(x,y)=\sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-j-1}{j}(ix)^{n-2j-1}y^j.$$

This completes the proof.

Theorem 1.2 (Asci & Gurel, 2012) *The bivariate complex Lucas polynomials is expressed explicitly as follows*

$$L_n(x,y)=\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{n}{n-j}\binom{n-j}{j}(ix)^{n-2j}y^j.$$

Proposition 1.3 (Boughaba *et al*., 2019) *For $n\in\mathbb{N}$, we have*

$$F_n(x,y)=h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right), \qquad (1)$$

and

$$L_n(x,y)=2h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)-ixh_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right). \qquad (2)$$

The rest of this section is devoted to recalling some preliminary facts and results on the symmetric functions.

Definition 1.4 (Boussayoud *et al*., 2021) *Let k and n be two positive integers and $\{a_1,a_2,...,a_n\}$ are set of given variables. The k-th elementary symmetric function $e_k(a_1,a_2,...,a_n)$ is defined by*

$$e_k(a_1,a_2,...,a_n)=\sum_{i_1+i_2+...+i_n=k}a_1^{i_1}a_2^{i_2}...a_n^{i_n}\qquad(0\le k\le n),$$

with $i_1,i_2,...,i_n=0$ or 1.

Definition 1.5 (Boussayoud *et al*., 2021) *Let k and n be two positive integers and $\{a_1,a_2,...,a_n\}$ are set of given variables. The k-th complete homogeneous symmetric function $h_k(a_1,a_2,...,a_n)$ is given by*

$$h_k(a_1,a_2,...,a_n)=\sum_{i_1+i_2+...+i_n=k}a_1^{i_1}a_2^{i_2}...a_n^{i_n}\qquad(k\ge 0),$$

with $i_1,i_2,...,i_n\ge 0$.

Remark 1.6 *Set $e_0(a_1,a_2,...,a_n)=1$ and $h_0(a_1,a_2,...,a_n)=1$, by usual convention. For $k<0$, we set $e_k(a_1,a_2,...,a_n)=0$ and $h_k(a_1,a_2,...,a_n)=0$.*

Definition 1.7 (Boussayoud & Boughaba, 2019) *Let* A and P *be any two alphabets. Let* $S_n(A-P)$ *be described by the following form*

$$\frac{\prod_{p\in P}(1-pz)}{\prod_{a\in A}(1-az)}=\sum_{n=0}^{\infty}S_n(A-P)z^n, \tag{3}$$

with the condition $S_n(A-P)=0$ *for* $n<0$.

Equation (3) can be rewritten in the following form

$$\sum_{n=0}^{\infty}S_n(A-P)z^n=\left(\sum_{n=0}^{\infty}S_n(A)z^n\right)\times\left(\sum_{n=0}^{\infty}S_n(-P)z^n\right),$$

where

$$S_n(A-P)=\sum_{j=0}^{n}S_{n-j}(-P)S_j(A).$$

Definition 1.8 (Boussayoud, 2017; Saba & Boussayoud, 2020) *Given a function* f *on* $\mathbb{R}^n$, *the divided difference operator is defined as follows*

$$\partial_{p_i p_{i+1}}(f)=\frac{\begin{pmatrix} f(p_1,\cdots,p_i,p_{i+1},\cdots,p_n)\\ -f(p_1,\cdots,p_{i-1},p_{i+1},p_i,p_{i+2},\cdots,p_n)\end{pmatrix}}{p_i-p_{i+1}}.$$

Definition 1.9 (Boussayoud & Abderrezzak, 2019) *Given an alphabet* $P=\{p_1,p_2\}$, *the symmetrizing operator* $\delta_{p_1p_2}^{k}$ *is defined by*

$$\delta_{p_1p_2}^{k}(f)=\frac{p_1^k f(p_1)-p_2^k f(p_2)}{p_1-p_2},\ \text{for all } k\in\mathbb{N}_0.$$

Remark 1.10 Let $k=0$ *and* $P=\{z,qz\}$ *in Definition 1.9, thus we have* (Purohit & Raina, 2015; Abderrezzak, 1994)

$$D_{q,z}f(z)=\frac{f(z)-f(qz)}{z-qz}\quad (z\neq 0,\ q\neq 1).$$

2. The generalized (*p*,*q*)-numbers

In this section, we introduce a new generalization of (*p*,*q*)-numbers.

Definition 2.1 *For any positive real numbers p and q, the sequence of generalized* (*p*,*q*)-*numbers* $\{W_{p,q,n}\}_{n\in\mathbb{N}}$ *is given by the following recurrence relation:*

$$W_{p,q,n}=apW_{p,q,n-1}+bqW_{p,q,n-2},\ (n\geq 2), \tag{4}$$

with $W_{p,q,0}=\alpha$, $W_{p,q,1}=\beta p+\gamma$ *and* $\{a,b,\alpha,\beta,\gamma\}\in\mathbb{N}$.

The special cases of the numbers $\{W_{p,q,n}\}_{n\in\mathbb{N}}$ are listed in the table below:

Table 1. (*p*,*q*)-numbers.

a	b	α	β	γ	$W_{p,q,n}$	(p,q)-numbers
1	1	0	0	1	$F_{p,q,n}$	(p,q)-Fibonacci numbers
1	1	2	1	0	$L_{p,q,n}$	(p,q)-Lucas numbers
1	2	0	0	1	$J_{p,q,n}$	(p,q)-Jacobsthal numbers
1	2	2	1	0	$j_{p,q,n}$	(p,q)-Jacobsthal Lucas numbers
2	1	0	0	1	$P_{p,q,n}$	(p,q)-Pell numbers
2	1	2	2	0	$Q_{p,q,n}$	(p,q)-Pell Lucas numbers

It is worth noting that the recurrence relationship in (4) involve the following characteristic equation

$$x^2-apx-bq=0,$$

which has two characteristic roots

$$x_1=\frac{ap+\sqrt{a^2p^2+4bq}}{2}\ \text{and}\ x_2=\frac{ap-\sqrt{a^2p^2+4bq}}{2},$$

and can verify the properties

$x_1+x_2=ap$, $x_1x_2=-bq$ and

$x_1-x_2=\sqrt{a^2p^2+4bq}$.

The next theorem gives the *n*-th term of the generalized (*p*,*q*)-numbers.

Theorem 2.2 *The Binet's formula for generalized* (*p*,*q*)-*numbers is given by*

$$W_{p,q,n}=\frac{Ax_1^n-Bx_2^n}{x_1-x_2}, \tag{5}$$

with $A=\beta p+\gamma-\alpha x_2$ *and* $B=\beta p+\gamma-\alpha x_1$.

Proof. According to the theory of difference equation, we have the following general term for generalized (*p*,*q*)-numbers

$$W_{p,q,n} = C_1 x_1^n + C_2 x_2^n,$$

where C_1 and C_2 are the coefficients.

For $n = 0,1,$ we have

$$\begin{cases} C_1 + C_2 = \alpha \\ C_1 x_1 + C_2 x_2 = \beta p + \gamma \end{cases}.$$

By these equalities

$$\begin{cases} C_1 = \frac{\beta p + \gamma - \alpha x_2}{x_1 - x_2} = \frac{A}{x_1 - x_2} \\ C_2 = \frac{\alpha x_1 - (\beta p + \gamma)}{x_1 - x_2} = -\frac{B}{x_1 - x_2} \end{cases}.$$

Therefore

$$W_{p,q,n} = \frac{A x_1^n - B x_2^n}{x_1 - x_2}.$$

This completes the proof.

The special cases of the Binet's formula for generalized (*p*,*q*)-numbers $\{W_{p,q,n}\}_{n\in\mathbb{N}}$ are listed in the table below:

Table 2. Binet's formulas of (*p*,*q*)-numbers.

a	b	α	β	γ	Roots (x_1 and x_2)	Binet's formula
1	1	0	0	1	$x_{1,2} = \frac{p\pm\sqrt{p^2+4q}}{2}$	$F_{p,q,n} = \frac{x_1^n - x_2^n}{x_1 - x_2}$
1	1	2	1	0	$x_{1,2} = \frac{p\pm\sqrt{p^2+4q}}{2}$	$L_{p,q,n} = x_1^n + x_2^n$
1	2	0	0	1	$x_{1,2} = \frac{p\pm\sqrt{p^2+8q}}{2}$	$J_{p,q,n} = \frac{x_1^n - x_2^n}{x_1 - x_2}$
1	2	2	1	0	$x_{1,2} = \frac{p\pm\sqrt{p^2+8q}}{2}$	$j_{p,q,n} = x_1^n + x_2^n$
2	1	0	0	1	$x_{1,2} = p \pm \sqrt{p^2+q}$	$P_{p,q,n} = \frac{x_1^n - x_2^n}{x_1 - x_2}$
2	1	2	2	0	$x_{1,2} = p \pm \sqrt{p^2+q}$	$Q_{p,q,n} = x_1^n + x_2^n$

3. Construction of generating functions of some numbers

First, let consider a proposition which will be of great interest to derive the main results. More details concerning this proposition are given in (Boussayoud & Kerada, 2014).

Proposition 3.1 *Given an alphabet* $A = \{a_1, a_2\}$, *then we have*

$$\sum_{n=0}^{\infty} h_n(a_1,a_2) z^n = \frac{1}{1-(a_1+a_2)z + a_1 a_2 z^2}. \quad (6)$$

Based on relationship (6), we have

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2) z^n = \frac{z}{1-(a_1+a_2)z + a_1 a_2 z^2}. \quad (7)$$

Choosing A such that $\begin{cases} a_1 = \frac{ap+\sqrt{a^2p^2+4bq}}{2} \\ a_2 = \frac{ap-\sqrt{a^2p^2+4bq}}{2} \end{cases}$

and substituting in (6) and (7), we obtain

$$\sum_{n=0}^{\infty} h_n\left(\frac{ap+\sqrt{a^2p^2+4bq}}{2}, \frac{ap-\sqrt{a^2p^2+4bq}}{2}\right) z^n = \frac{1}{1-apz-bqz^2}, \quad (8)$$

$$\sum_{n=0}^{\infty} h_{n-1}\left(\frac{ap+\sqrt{a^2p^2+4bq}}{2}, \frac{ap-\sqrt{a^2p^2+4bq}}{2}\right) z^n = \frac{z}{1-apz-bqz^2}, \quad (9)$$

respectively. Multiplying equation (8) by (α) and adding it to the equation obtained by (9) and multiplying by $\left(p\left(\beta-\alpha a\right)+\gamma\right)$, then we obtain

$$\sum_{n=0}^{\infty} \left(\begin{array}{c} \alpha h_n\left(\frac{ap+\sqrt{a^2p^2+4bq}}{2}, \frac{ap-\sqrt{a^2p^2+4bq}}{2}\right) \\ +\left(p\left(\beta-\alpha a\right)+\gamma\right) h_{n-1}\left(\frac{ap+\sqrt{a^2p^2+4bq}}{2}, \frac{ap-\sqrt{a^2p^2+4bq}}{2}\right) \end{array} \right) z^n$$

$$= \frac{\alpha + \left(p\left(\beta-\alpha a\right)+\gamma\right)z}{1-apz-bqz^2},$$

Consequently, we have the following theorem.

Theorem 3.2 A generating function is derived for the generalized (p,q)-numbers as follows

$$\sum_{n=0}^{\infty} W_{p,q,n} z^n = \frac{\alpha + \left(p\left(\beta-\alpha a\right)+\gamma\right)z}{1-apz-bqz^2}, \quad (10)$$

with

$$W_{p,q,n} = \alpha h_n\left(\frac{ap+\sqrt{a^2p^2+4bq}}{2},\frac{ap-\sqrt{a^2p^2+4bq}}{2}\right)$$
$$+\left(p\left(\beta-\alpha a\right)+\gamma\right)h_{n-1}\left(\frac{ap+\sqrt{a^2p^2+4bq}}{2},\frac{ap-\sqrt{a^2p^2+4bq}}{2}\right).$$

Proof. The generalized (p,q)-numbers can be considered as the coefficients of the following formal power series

$$g(z) = \sum_{n=0}^{\infty} W_{p,q,n} z^n.$$

Using the initial conditions, we get

$$\begin{aligned} g(z) &= W_{p,q,0} + W_{p,q,1}z + \sum_{n=2}^{\infty} W_{p,q,n} z^n \\ &= W_{p,q,0} + W_{p,q,1}z \\ &\quad + \sum_{n=2}^{\infty}\left(apW_{p,q,n-1} + bqW_{p,q,n-2}\right)z^n \\ &= W_{p,q,0} + W_{p,q,1}z + apz\sum_{n=1}^{\infty} W_{p,q,n} z^n \\ &\quad + bqz^2\sum_{n=0}^{\infty} W_{p,q,n} z^n \\ &= W_{p,q,0} + \left(W_{p,q,1} - apW_{p,q,0}\right)z \\ &\quad + apz\sum_{n=0}^{\infty} W_{p,q,n} z^n + bqz^2\sum_{n=0}^{\infty} W_{p,q,n} z^n \\ &= \alpha + \left(p\left(\beta-\alpha a\right)+\gamma\right)z + \left(apz + bqz^2\right)g(z). \end{aligned}$$

Hence, we obtain

$$\left(1-apz-bqz^2\right)g(z) = \alpha + \left(p\left(\beta-\alpha a\right)+\gamma\right)z.$$

Therefore

$$g(z) = \frac{\alpha + \left(p\left(\beta-\alpha a\right)+\gamma\right)z}{1-apz-bqz^2}.$$

The proof is completed.

- By setting $a = b = \gamma = 1$ and $\alpha = \beta = 0$ in expression (10), we can state the following corollary.

Corollary 3.3 *A generating function is derived for the (p,q)-Fibonacci numbers as follows*

$$\sum_{n=0}^{\infty} F_{p,q,n} z^n = \frac{z}{1-pz-qz^2}, \quad (11)$$

with

$$F_{p,q,n} = h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right).$$

- Put $p = q = 1$ in (11) we get the generating function of Fibonacci numbers

$$\sum_{n=0}^{\infty} F_n z^n = \frac{z}{1-z-z^2},$$

with

$$F_n = h_{n-1}\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}\right).$$

- By putting $a = b = \beta = 1$, $\alpha = 2$ and $\gamma = 0$ in expression (10), we can state the following corollary.

Corollary 3.4 *For* $n \in \mathbb{N}$, *the new generating function of* (p,q)*-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} L_{p,q,n} z^n = \frac{2-pz}{1-pz-qz^2}, \quad (12)$$

with

$$L_{p,q,n} = 2h_n\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right)$$
$$-ph_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right).$$

- Let $p = q = 1$ in expression (12) we derive the generating function of Lucas numbers

$$\sum_{n=0}^{\infty} L_n z^n = \frac{2-z}{1-z-z^2},$$

with

$$L_n = 2h_n\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}\right)$$
$$-h_{n-1}\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2}\right).$$

- By taking $a = \gamma = 1$, $b = 2$ and $\alpha = \beta = 0$ in expression (10), we can state the following corollary.

Corollary 3.5 *A generating function is obtained for the (p,q)- Jacobsthal numbers as follows*

$$\sum_{n=0}^{\infty} J_{p,q,n} z^n = \frac{z}{1 - pz - 2qz^2}, \qquad (13)$$

with

$$J_{p,q,n} = h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right).$$

- Let $p = q = 1$ in expression (13) we easily derive the generating function of Jacobsthal numbers

$$\sum_{n=0}^{\infty} J_n z^n = \frac{z}{1 - z - 2z^2},$$

with

$$J_n = h_{n-1}(2,-1).$$

- By setting $a = \beta = 1,\ b = \alpha = 2$ and $\gamma = 0$ in expression (10), we can state the following corollary.

Corollary 3.6 *A generating function is obtained for the (p,q)-Jacobsthal Lucas numbers by the following expression*

$$\sum_{n=0}^{\infty} j_{p,q,n} z^n = \frac{2 - pz}{1 - pz - 2qz^2}, \qquad (14)$$

with

$$j_{p,q,n} = 2h_n\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) - ph_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right).$$

- Put $p = q = 1$ in relation (14) results in the following generating function of Jacobsthal Lucas numbers

$$\sum_{n=0}^{\infty} j_n z^n = \frac{2 - z}{1 - z - 2z^2},$$

with

$$j_n = 2h_n(2,-1) - h_{n-1}(2,-1).$$

- By putting $a = 2,\ b = \gamma = 1$ and $\alpha = \beta = 0$ in expression (10), we can state the following corollary.

Corollary 3.7 *A generating function is obtained for the (p,q)- Pell numbers by the following expression*

$$\sum_{n=0}^{\infty} P_{p,q,n} z^n = \frac{z}{1 - 2pz - qz^2}, \qquad (15)$$

with

$$P_{p,q,n} = h_{n-1}\left(p + \sqrt{p^2+q}, p - \sqrt{p^2+q}\right).$$

- Put $p = q = 1$ in expression (15) allows to obtain the generating function of Pell numbers

$$\sum_{n=0}^{\infty} P_n z^n = \frac{z}{1 - 2z - z^2},$$

with

$$P_n = h_{n-1}\left(1+\sqrt{2}, 1-\sqrt{2}\right).$$

- By taking $a = \alpha = \beta = 2,\ b = 1$ and $\gamma = 0$ in expression (10), we derive the following corollary.

Corollary 3.8 *A generating function is obtained for the (p,q)- Pell Lucas numbers as follows*

$$\sum_{n=0}^{\infty} Q_{p,q,n} z^n = \frac{2 - 2pz}{1 - 2pz - qz^2}, \qquad (16)$$

with

$$Q_{p,q,n} = 2h_n\left(p + \sqrt{p^2+q}, p - \sqrt{p^2+q}\right) - 2ph_{n-1}\left(p + \sqrt{p^2+q}, p - \sqrt{p^2+q}\right).$$

- Let $p = q = 1$ in expression (16) then the generating function of Pell Lucas numbers is given by

$$\sum_{n=0}^{\infty} Q_n z^n = \frac{2-2z}{1-2z-z^2},$$

with

$$Q_n = 2h_n\left(1+\sqrt{2},1-\sqrt{2}\right) - 2h_{n-1}\left(1+\sqrt{2},1-\sqrt{2}\right).$$

4. Generating functions of the products of (*p,q*)-numbers with bivariate complex Fibonacci and Lucas polynomials

The following propositions, proved in (Boussayoud, 2017; Boussayoud & Kerada, 2014; Boussayoud *et al*., 2014), are key tools to derive our main result.

Proposition 4.1 *Given two alphabets* $E=\{e_1,e_2\}$ *and* $A=\{a_1,a_2\}$, *then we have*

$$\sum_{n=0}^{\infty} h_n(a_1,a_2)h_n(e_1,e_2)z^n = \frac{1-a_1a_2e_1e_2z^2}{\prod_{i=1}^{2}(1-a_ie_1z)\prod_{i=1}^{2}(1-a_ie_2z)}. \quad (17)$$

Accordingly, we obtain

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2)h_{n-1}(e_1,e_2)z^n = \frac{z-a_1a_2e_1e_2z^3}{\prod_{i=1}^{2}(1-a_ie_1z)\prod_{i=1}^{2}(1-a_ie_2z)}. \quad (18)$$

Proposition 4.2 *Given two alphabets* $E=\{e_1,e_2\}$ *and* $A=\{a_1,a_2\}$, *we have*

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,a_2)h_n(e_1,e_2)z^n = \frac{(e_1+e_2)z-e_1e_2(a_1+a_2)z^2}{\prod_{i=1}^{2}(1-a_ie_1z)\prod_{i=1}^{2}(1-a_ie_2z)}. \quad (19)$$

Proposition 4.3 *Given two alphabets* $E=\{e_1,e_2\}$ *and* $A=\{a_1,a_2\}$, *we have*

$$\sum_{n=0}^{\infty} h_n(a_1,a_2)h_{n-1}(e_1,e_2)z^n = \frac{(a_1+a_2)z-a_1a_2(e_1+e_2)z^2}{\prod_{i=1}^{2}(1-a_ie_1z)\prod_{i=1}^{2}(1-a_ie_2z)}. \quad (20)$$

In what follows, we provide new theorems and derive the new generating functions for the products of (*p,q*)-Fibonacci numbers, (*p,q*)-Lucas numbers, (*p,q*)-Pell numbers, (*p,q*)-Pell Lucas numbers, (*p,q*)-Jacobsthal numbers and (*p,q*)-Jacobsthal Lucas numbers with bivariate complex Fibonacci and Lucas polynomials.

This part consists of three cases.

Case 1. Putting $A=\left\{\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right\}$ and $E=\left\{\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right\}$, in expression (17), (18), (19) and (20) then we have

$$\sum_{n=0}^{\infty}\left(\begin{array}{c} h_n\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{array}\right)z^n = \frac{L_1}{D_1}, \quad (21)$$

$$\sum_{n=0}^{\infty}\left(\begin{array}{c} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{array}\right)z^n = \frac{M_1}{D_1}, \quad (22)$$

$$\sum_{n=0}^{\infty}\left(\begin{array}{c} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{array}\right)z^n = \frac{N_1}{D_1}, \quad (23)$$

$$\sum_{n=0}^{\infty}\left(\begin{array}{c} h_n\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{array}\right)z^n = \frac{R_1}{D_1}, \quad (24)$$

with

$$D_1 = 1-ipxz-\left(y\left(p^2+2q\right)-qx^2\right)z^2 - ipqxyz^3+q^2y^2z^4,$$

$$L_1 = 1-qyz^2,$$

$$M_1 = z-qyz^3,$$

$$N_1 = ixz+pyz^2,$$

$$R_1 = pz+iqxz^2,$$

Accordingly, we deduce the following corollary and theorems.

Corollary 4.4 *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} F_{p,q,n} F_n(x,y) z^n = \frac{M_1}{D_1} = \frac{z - qyz^3}{1 - ipxz - \left(y\left(p^2+2q\right)-qx^2\right)z^2 - ipqxyz^3 + q^2y^2z^4}, \tag{25}$$

with

$$F_{p,q,n}F_n(x,y) = h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right).$$

Theorem 4.5 *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci numbers with bivariate complex Lucas polynomials is expressed as follows*

$$\sum_{n=0}^{\infty} F_{p,q,n} L_n(x,y) z^n = \frac{ixz + 2pyz^2 + iqxyz^3}{1 - ipxz - \left(y\left(p^2+2q\right)-qx^2\right)z^2 - ipqxyz^3 + q^2y^2z^4}. \tag{26}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{p,q,n} L_n(x,y) z^n
&= \sum_{n=0}^{\infty} \left(\begin{array}{l} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times \left(\begin{array}{l} 2h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \\ -ixh_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{array} \right) \end{array} \right) z^n \\
&= 2\sum_{n=0}^{\infty} \left(\begin{array}{l} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{array} \right) z^n \\
&\quad - ix\sum_{n=0}^{\infty} \left(\begin{array}{l} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{array} \right) z^n \\
&= \frac{2N_1 - ixM_1}{D_1} \\
&= \frac{2\left(ixz + pyz^2\right) - ix\left(z - qyz^3\right)}{1 - ipxz - \left(y\left(p^2+2q\right)-qx^2\right)z^2 - ipqxyz^3 + q^2y^2z^4} \\
&= \frac{ixz + 2pyz^2 + iqxyz^3}{1 - ipxz - \left(y\left(p^2+2q\right)-qx^2\right)z^2 - ipqxyz^3 + q^2y^2z^4}.
\end{aligned}$$

This completes the proof.

Theorem 4.6 *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Lucas numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} L_{p,q,n} F_n(x,y) z^n = \frac{pz + 2iqxz^2 + pqyz^3}{1 - ipxz - \left(y\left(p^2+2q\right)-qx^2\right)z^2 - ipqxyz^3 + q^2y^2z^4}. \tag{27}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{p,q,n} F_n(x,y) z^n
&= \sum_{n=0}^{\infty} \left(\begin{array}{l} \left(\begin{array}{l} 2h_n\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ -ph_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \end{array} \right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{array} \right) z^n \\
&= 2\sum_{n=0}^{\infty} \left(\begin{array}{l} h_n\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{array} \right) z^n \\
&\quad - p\sum_{n=0}^{\infty} \left(\begin{array}{l} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2}, \frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{array} \right) z^n \\
&= \frac{2R_1 - pM_1}{D_1} \\
&= \frac{2\left(pz + iqxz^2\right) - p\left(z - qyz^3\right)}{1 - ipxz - \left(y\left(p^2+2q\right)-qx^2\right)z^2 - ipqxyz^3 + q^2y^2z^4} \\
&= \frac{pz + 2iqxz^2 + pqyz^3}{1 - ipxz - \left(y\left(p^2+2q\right)-qx^2\right)z^2 - ipqxyz^3 + q^2y^2z^4}.
\end{aligned}$$

This completes the proof.

Theorem 4.7 *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Lucas numbers with bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} L_{p,q,n} L_n(x,y) z^n = \frac{4-3ipxz+2\left(qx^2-y\left(p^2+2q\right)\right)z^2-ipqxyz^3}{1-ipxz-\left(y\left(p^2+2q\right)-qx^2\right)z^2-ipqxyz^3+q^2y^2z^4}. \quad (28)$$

Proof. We have

$$\sum_{n=0}^{\infty} L_{p,q,n} L_n(x,y) z^n = \sum_{n=0}^{\infty} \left(\begin{pmatrix} 2h_n\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ -ph_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \end{pmatrix} \times \begin{pmatrix} 2h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right) \\ -ixh_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{pmatrix} \right) z^n$$

$$= 4\sum_{n=0}^{\infty} \begin{pmatrix} h_n\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{pmatrix} z^n$$

$$-2ix\sum_{n=0}^{\infty} \begin{pmatrix} h_n\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{pmatrix} z^n$$

$$-2p\sum_{n=0}^{\infty} \begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{pmatrix} z^n$$

$$+ipx\sum_{n=0}^{\infty} \begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+4q}}{2},\frac{p-\sqrt{p^2+4q}}{2}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{pmatrix} z^n$$

$$= \frac{4L_1-2ixR_1-2pN_1+ipxM_1}{D_1}$$

$$= \frac{4\left(1-qyz^2\right)-2ix\left(pz+iqxz^2\right)}{1-ipxz-\left(y\left(p^2+2q\right)-qx^2\right)z^2-ipqxyz^3+q^2y^2z^4}$$

$$-\frac{2p\left(ixz+pyz^2\right)-ipx\left(z-qyz^3\right)}{1-ipxz-\left(y\left(p^2+2q\right)-qx^2\right)z^2-ipqxyz^3+q^2y^2z^4}$$

$$= \frac{4-3ipxz+2\left(qx^2-y\left(p^2+2q\right)\right)z^2-ipqxyz^3}{1-ipxz-\left(y\left(p^2+2q\right)-qx^2\right)z^2-ipqxyz^3+q^2y^2z^4}.$$

This completes the proof.

- Put $p=q=1$ in expression (25), (26), (27) and (28), we obtain the following table:

Table 3. The generating functions of the products of Fibonacci and Lucas numbers with bivariate complex polynomials.

Coefficient of z^n	Generating function
$F_n F_n(x,y)$	$\frac{z-yz^3}{1-ixz-\left(3y-x^2\right)z^2-ixyz^3+y^2z^4}$
$F_n L_n(x,y)$	$\frac{ixz+2yz^2+ixyz^3}{1-ixz-\left(3y-x^2\right)z^2-ixyz^3+y^2z^4}$
$L_n F_n(x,y)$	$\frac{z+2ixz^2+yz^3}{1-ixz-\left(3y-x^2\right)z^2-ixyz^3+y^2z^4}$
$L_n L_n(x,y)$	$\frac{4-3ixz+2\left(x^2-3y\right)z^2-ixyz^3}{1-ixz-\left(3y-x^2\right)z^2-ixyz^3+y^2z^4}$

Case 2. For $A=\left\{p+\sqrt{p^2+q},p-\sqrt{p^2+q}\right\}$ and $E=\left\{\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right\}$, in the relationships (17), (18), (19) and (20) we have

$$\sum_{n=0}^{\infty} \begin{pmatrix} h_n\left(p+\sqrt{p^2+q},p-\sqrt{p^2+q}\right) \\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{pmatrix} z^n = \frac{L_2}{D_2}, \quad (29)$$

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n=\frac{M_2}{D_2}, \quad (30)$$

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_{n}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n=\frac{N_2}{D_2}, \quad (31)$$

$$\sum_{n=0}^{\infty}\begin{pmatrix} h_{n}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n=\frac{R_2}{D_2}, \quad (32)$$

with

$$D_2 = 1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2 -2ipqxyz^3+q^2y^2z^4,$$

$$L_2 = 1-qyz^2,$$

$$M_2 = z-qyz^3,$$

$$N_2 = ixz+2pyz^2,$$

$$R_2 = 2pz+iqxz^2,$$

and we deduce the following corollary and theorems.

Corollary 4.8 *For* $n \in \mathbb{N}$, *the new generating function of the product of (p,q)-Pell numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} P_{p,q,n}F_n(x,y)z^n=\frac{M_2}{D_2} = \frac{z-qyz^3}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}, \quad (33)$$

with

$$P_{p,q,n}F_n(x,y)=h_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right).$$

Theorem 4.9 *For* $n \in \mathbb{N}$, *the new generating function of the product of (p,q)-Pell numbers with bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} P_{p,q,n}L_n(x,y)z^n = \frac{ixz+4pyz^2+iqxyz^3}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}. \quad (34)$$

Proof. We have

$$\sum_{n=0}^{\infty} P_{p,q,n}L_n(x,y)z^n$$

$$=\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times\begin{pmatrix} 2h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right) \\ -ixh_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix}\end{pmatrix} z^n$$

$$=2\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_{n}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$-ix\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$=\frac{2N_2-ixM_2}{D_2}$$

$$=\frac{2\left(ixz+2pyz^2\right)-ix\left(z-qyz^3\right)}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}$$

$$=\frac{ixz+4pyz^2+iqxyz^3}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}.$$

This completes the proof.

Theorem 4.10 *For* $n \in \mathbb{N}$, *the new generating function of the product of (p,q)-Pell Lucas numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} Q_{p,q,n}F_n(x,y)z^n = \frac{2pz+2iqxz^2+2pqyz^3}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}. \quad (35)$$

Proof. We have

$$\sum_{n=0}^{\infty} Q_{p,q,n} F_n(x,y) z^n$$

$$=\sum_{n=0}^{\infty}\left(\begin{pmatrix} 2h_n\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ -2ph_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right)\end{pmatrix} \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\right) z^n$$

$$=2\sum_{n=0}^{\infty}\begin{pmatrix} h_n\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$-2p\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$=\frac{2R_2-2pM_2}{D_2}$$

$$=\frac{2\left(2pz+iqxz^2\right)-2p\left(z-qyz^3\right)}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}$$

$$=\frac{2pz+2iqxz^2+2pqyz^3}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}.$$

This completes the proof.

Theorem 4.11 *For $n\in\mathbb{N}$, the new generating function of the product of* (*p*,*q*)*-Pell Lucas numbers with bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} Q_{p,q,n} L_n(x,y) z^n = \frac{4-6ipxz+2\left(qx^2-2y\left(2p^2+q\right)\right)z^2-2ipqxyz^3}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}. \quad (36)$$

Proof. We have

$$\sum_{n=0}^{\infty} Q_{p,q,n} L_n(x,y) z^n$$

$$=\sum_{n=0}^{\infty}\left(\begin{pmatrix} 2h_n\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ -2ph_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right)\end{pmatrix} \\ \times\begin{pmatrix} 2h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right) \\ -ixh_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix}\right) z^n$$

$$=4\sum_{n=0}^{\infty}\begin{pmatrix} h_n\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$-2ix\sum_{n=0}^{\infty}\begin{pmatrix} h_n\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$-4p\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$+2ipx\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(p+\sqrt{p^2+q},\, p-\sqrt{p^2+q}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$=\frac{4L_2-2ixR_2-4pN_2+2ipxM_2}{D_2}$$

$$=\frac{4\left(1-qyz^2\right)-2ix\left(2pz+iqxz^2\right)}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}$$

$$-\frac{4p\left(ixz+2pyz^2\right)-2ipx\left(z-qyz^3\right)}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}$$

$$=\frac{4-6ipxz+2\left(qx^2-2y\left(2p^2+q\right)\right)z^2-2ipqxyz^3}{1-2ipxz-\left(2y\left(2p^2+q\right)-qx^2\right)z^2-2ipqxyz^3+q^2y^2z^4}.$$

This completes the proof.

- Put $p=q=1$ in expression (33), (34), (35) and (36), we obtain the following table:

Table 4. The generating functions of the products of Pell and Pell Lucas numbers with bivariate complex polynomials.

Coefficient of z^n	Generating function
$P_n F_n(x,y)$	$\frac{z-yz^3}{1-2ixz-\left(6y-x^2\right)z^2-2ixyz^3+y^2z^4}$
$P_n L_n(x,y)$	$\frac{ixz+4yz^2+ixyz^3}{1-2ixz-\left(6y-x^2\right)z^2-2ixyz^3+y^2z^4}$
$Q_n F_n(x,y)$	$\frac{2z+2ixz^2+2yz^3}{1-2ixz-\left(6y-x^2\right)z^2-2ixyz^3+y^2z^4}$
$Q_n L_n(x,y)$	$\frac{4-6ixz+2\left(x^2-6y\right)z^2-2ixyz^3}{1-2ixz-\left(6y-x^2\right)z^2-2ixyz^3+y^2z^4}$

Case 3. For $A=\left\{\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right\}$ and $E=\left\{\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right\}$, in expression (17), (18), (19) and (20) we have

$$\sum_{n=0}^{\infty}\left(\begin{array}{l}h_n\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{array}\right)z^n=\frac{L_3}{D_3}, \quad (37)$$

$$\sum_{n=0}^{\infty}\left(\begin{array}{l}h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{array}\right)z^n=\frac{M_3}{D_3}, \quad (38)$$

$$\sum_{n=0}^{\infty}\left(\begin{array}{l}h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{array}\right)z^n=\frac{N_3}{D_3}, \quad (39)$$

$$\sum_{n=0}^{\infty}\left(\begin{array}{l}h_n\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{array}\right)z^n=\frac{R_3}{D_3}, \quad (40)$$

with

$$D_3=1-ipxz-\left(y\left(p^2+4q\right)-2qx^2\right)z^2$$
$$-2ipqxyz^3+4q^2y^2z^4,$$
$$L_3=1-2qyz^2,$$
$$M_3=z-2qyz^3,$$
$$N_3=ixz+pyz^2,$$
$$R_3=pz+2iqxz^2,$$

and we deduce the following corollary and theorems.

Corollary 4.12 *For $n\in\mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal numbers with bivariate complex Fibonacci polynomials is defined by*

$$\sum_{n=0}^{\infty}J_{p,q,n}F_n(x,y)z^n=\frac{M_3}{D_3}$$
$$=\frac{z-2qyz^3}{1-ipxz-\left(y\left(p^2+4q\right)-2qx^2\right)z^2-2ipqxyz^3+4q^2y^2z^4}, \quad (41)$$

with

$$J_{p,q,n}F_n(x,y)=h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)$$
$$\times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right).$$

Theorem 4.13 *For $n\in\mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal numbers with bivariate complex Lucas polynomials is expressed as*

$$\sum_{n=0}^{\infty}J_{p,q,n}L_n(x,y)z^n$$
$$=\frac{ixz+2pyz^2+2iqxyz^3}{1-ipxz-\left(y\left(p^2+4q\right)-2qx^2\right)z^2-2ipqxyz^3+4q^2y^2z^4}. \quad (42)$$

Proof. We have

$$\sum_{n=0}^{\infty} J_{p,q,n} L_n(x,y) z^n$$

$$= \sum_{n=0}^{\infty} \left(h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \times \begin{pmatrix} 2h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \\ -ixh_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \end{pmatrix} \right) z^n$$

$$= 2\sum_{n=0}^{\infty} \left(h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \right) z^n$$

$$-ix\sum_{n=0}^{\infty} \left(h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \right) z^n$$

$$= \frac{2N_3 - ixM_3}{D_3}$$

$$= \frac{2\left(ixz + pyz^2\right) - ix\left(z - 2qyz^3\right)}{1 - ipxz - \left(y\left(p^2+4q\right) - 2qx^2\right)z^2 - 2ipqxyz^3 + 4q^2y^2z^4}$$

$$= \frac{ixz + 2pyz^2 + 2iqxyz^3}{1 - ipxz - \left(y\left(p^2+4q\right) - 2qx^2\right)z^2 - 2ipqxyz^3 + 4q^2y^2z^4}.$$

This completes the proof.

Theorem 4.14 *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal Lucas numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} j_{p,q,n} F_n(x,y) z^n = \frac{pz + 4iqxz^2 + 2pqyz^3}{1 - ipxz - \left(y\left(p^2+4q\right) - 2qx^2\right)z^2 - 2ipqxyz^3 + 4q^2y^2z^4}. \tag{43}$$

Proof. We have

$$\sum_{n=0}^{\infty} j_{p,q,n} F_n(x,y) z^n$$

$$= \sum_{n=0}^{\infty} \left(\begin{pmatrix} 2h_n\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \\ -ph_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \end{pmatrix} \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \right) z^n$$

$$= 2\sum_{n=0}^{\infty} \left(h_n\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \right) z^n$$

$$-p\sum_{n=0}^{\infty} \left(h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2}, \frac{p-\sqrt{p^2+8q}}{2}\right) \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2}, \frac{ix-\sqrt{-x^2+4y}}{2}\right) \right) z^n$$

$$= \frac{2R_3 - pM_3}{D_3}$$

$$= \frac{2\left(pz + 2iqxz^2\right) - p\left(z - 2qyz^3\right)}{1 - ipxz - \left(y\left(p^2+4q\right) - 2qx^2\right)z^2 - 2ipqxyz^3 + 4q^2y^2z^4}$$

$$= \frac{pz + 4iqxz^2 + 2pqyz^3}{1 - ipxz - \left(y\left(p^2+4q\right) - 2qx^2\right)z^2 - 2ipqxyz^3 + 4q^2y^2z^4}.$$

This completes the proof.

Theorem 4.15 *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Jacobsthal Lucas numbers with bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} j_{p,q,n} L_n(x,y) z^n = \frac{4 - 3ipxz + 2\left(2qx^2 - y\left(p^2+4q\right)\right)z^2 - 2ipqxyz^3}{1 - ipxz - \left(y\left(p^2+4q\right) - 2qx^2\right)z^2 - 2ipqxyz^3 + 4q^2y^2z^4}. \tag{44}$$

Proof. We have

$$\sum_{n=0}^{\infty} j_{p,q,n} L_n(x,y) z^n$$

$$=\sum_{n=0}^{\infty}\left(\begin{pmatrix} 2h_n\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right) \\ -ph_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right)\end{pmatrix} \times \begin{pmatrix} 2h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right) \\ -ixh_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix}\right) z^n$$

$$=4\sum_{n=0}^{\infty}\begin{pmatrix} h_n\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$-2ix\sum_{n=0}^{\infty}\begin{pmatrix} h_n\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$-2p\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_n\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$+ipx\sum_{n=0}^{\infty}\begin{pmatrix} h_{n-1}\left(\frac{p+\sqrt{p^2+8q}}{2},\frac{p-\sqrt{p^2+8q}}{2}\right) \\ \times h_{n-1}\left(\frac{ix+\sqrt{-x^2+4y}}{2},\frac{ix-\sqrt{-x^2+4y}}{2}\right)\end{pmatrix} z^n$$

$$=\frac{4L_3-2ixR_3-2pN_3+ipxM_3}{D_3}$$

$$=\frac{4\left(1-2qyz^2\right)-2ix\left(pz+2iqxz^2\right)}{1-ipxz-\left(y\left(p^2+4q\right)-2qx^2\right)z^2-2ipqxyz^3+4q^2y^2z^4}$$

$$-\frac{2p\left(ixz+pyz^2\right)-ipx\left(z-2qyz^3\right)}{1-ipxz-\left(y\left(p^2+4q\right)-2qx^2\right)z^2-2ipqxyz^3+4q^2y^2z^4}$$

$$=\frac{4-3ipxz+2\left(2qx^2-y\left(p^2+4q\right)\right)z^2-2ipqxyz^3}{1-ipxz-\left(y\left(p^2+4q\right)-2qx^2\right)z^2-2ipqxyz^3+4q^2y^2z^4}.$$

This completes the proof.

- Put $p=q=1$ in expression (41), (42), (43) and (44), we obtain the following table:

Table 5. The generating functions of the products of Jacobsthal and Jacobsthal Lucas numbers with bivariate complex polynomials.

Coefficient of z^n	Generating function
$J_nF_n(x,y)$	$\frac{z-2yz^3}{1-ixz-\left(5y-2x^2\right)z^2-2ixyz^3+4y^2z^4}$
$J_nL_n(x,y)$	$\frac{ixz+2yz^2+2ixyz^3}{1-ixz-\left(5y-2x^2\right)z^2-2ixyz^3+4y^2z^4}$
$j_nF_n(x,y)$	$\frac{z+4ixz^2+2yz^3}{1-ixz-\left(5y-2x^2\right)z^2-2ixyz^3+4y^2z^4}$
$j_nL_n(x,y)$	$\frac{4-3ixz+2\left(2x^2-5y\right)z^2-2ixyz^3}{1-ixz-\left(5y-2x^2\right)z^2-2ixyz^3+4y^2z^4}$

4. Conclusion

In this paper, a new generalization of (p,q)-numbers and new theorems have been proposed to determine the generating functions. The proposed theorems are based on symmetric functions.

In our forthcoming investigation, we plan to establish further results and properties associated with some generalized forms of the above-mentioned families of numbers.

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