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On the least trimmed squares estimators for *JS* circular regression model

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Abstract

The least trimmed squares (*LTS*) estimation has been successfully used in the robust linear regression models. This article extends the *LTS* estimation to the Jammalamadaka and Sarma (*JS*) circular regression model. The robustness of the proposed estimator is studied and the used algorithm for computation is discussed. Simulation studied, and real data show that the proposed robust circular estimator effectively fits *JS* circular models in the presence of vertical outliers and leverage points.

Keywords: Breakdown point; circular regression; *LTS* estimation; outliers; robust estimation.

1. Introduction

Circular data or directional data are having con- siderable broadly used in different areas such as natural science e.g: (Rivest, 1997), medical sciences e.g: (Downs & Mardia, 2002), meterology e.g: (Kato *et al*., 2008), biology e.g: (Lund, 1999) and geophysics e.g: (Chang *et al*., 1990). Strong interests in circular regression model have also been shown see, (Gould, 1969; Mardia, 1975; Laycock, 2003; Down *et al*., 1971) and (Hussin *et al*., 2004). Another model of our interest is proposed by (Sarma & Jammalamadaka, 1993) when both response variable v and explanatory variable u are circular. They used the conditional expectation of the vector e^{iv} given u to represent the relationship between v and u. The properties of the models for the case of a single explanatory variable have been studied see, Section. 8.6 of (Jammalamadaka *et al*., 2001). (Ibrahim, 2013) extended the model by introducing p circular explanatory variables in the model and studied its ridge-estimators (Asar & Genc¸, 2017).

The problem of outliers in the circular regres- sion context has been well discussed. Most of the outliers' detection procedures were derived based on the simple circular regression model (Hussin *et al*., 2004) by extending the common methods from linear regression (Abuzaid *et al*., 2008) and (Abuzaid *et al*., 2013). This model assumed a linear relationship between the two circular variables, which is a conservative condition. Detection of outliers in *JS* model was considered in (Ibrahim *et al*., 2013), where they used the *COV RATIO* statistic to identify the outliers. Moreover, (Alkasadi *et al*., 2018) derived an outlier detection procedure for the multiple *JS* model with two independent circular variables.

Recently, (Jha & Biswas, 2017) becomes the first paper to consider the robust estimation for the (Kato *et al*., 2008) circular regression model based on wrapped Cauchy distribution settings by proposing the maximum trimmed cosine estimator. There is no published work on the robust estimation of *JS* circular regression as far our knowledge goes.

The trimming techniques introduced by (Rousseeuw, 1985) used in robust linear regression modeling, and several studies were used this estimator to overcome the problem of outliers instead of the classical estimator and illustrated excellent performance in linear regression models see, (Saleh, 2014). This article considers the *JS* circular regression model of (Sarma &

Jammalamadaka, 1993) which is known to have very interesting properties closely related to the theory of multiple linear regression and obtain the robust estimation using least trimmed square residual approach as mentioned in (Rousseeuw, 1985).

The rest of the article is organized as follows: Section 2 reviews the formulation of the *JS* circular regression model, and its parameters estimates methods. Section 3 formulates the effect of outliers in the *JS* model and discusses the robustness problem of this model based on the breakdown point. Section 4, the circular least trimmed squares (*CLTS*) estimator is defined, discusses its breakdown point and computational algorithm. A simulation study is conducted to study the performance of the proposed estimator in Section 5. Section 6 applies the robust estima- tors to the eye data set.

2. The *JS* circular regression model

2.1. Model formulation

For any two circular random variables **U** and **V**, (Sarma & Jammalamadaka, 1993) proposed a regression model to predict *v* for a given *u*, consider the conditional expectation of the vector eiv given u such that

$$E(e^{iv}|u) = \rho(u)e^{i\mu(u)} = g_1(u) + ig_2(u), \quad (1)$$

where, $e^{iv} = \cos v + i \sin v$, $\mu(u)$ represents the conditional mean direction of v given u and $\rho(u)$ represents the conditional concentration parameter. Equivalently, we may write

$$E(\cos v|u) = g_1(u) \quad \text{and} \quad E(\sin v|u) = g_2(u). \quad (2)$$

Then, v can be predicted such that

$$\mu(u) = \hat{v} = \arctan\frac{g_2(u)}{g_1(u)}$$

$$= \begin{cases} \arctan\frac{g_2(u)}{g_1(u)} & \text{if } g_1(u) \geq 0, \\ \pi + \arctan\frac{g_2(u)}{g_1(u)} & \text{if } g_1(u) \leq 0, \\ undefined & \text{if } g_1(u) = g_2(u) = 0. \end{cases} \quad (3)$$

Because $g_1(u)$ and $g_2(u)$ are periodic functions, thus they are approximated for a suitable degree m (Kufner, 1971), which have the following two observational regression-like models

$$V_{1j} = \cos v_j = g_1(u)$$

$$\simeq \sum_{k=0}^{m} (A_k \cos ku_j + B_k \sin ku_j) + \varepsilon_{1j}$$

and

$$V_{2j} = \sin v_j = g_2(u)$$

$$\simeq \sum_{k=0}^{m} (C_k \cos ku_j + D_k \sin ku_j) + \varepsilon_{2j}, \quad (4)$$

for $j = 1, ..., n$, where, $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is the vector of random errors following the bivariate normal distribution with mean vector 0 and un- known dispersion matrix Σ. The parameters A_k, B_k, C_k and D_k, $k = 0, 1, ..., m$, the standard errors as well as the matrix Σ can then be esti- mated. Assume that $B_0 = D_0 = 0$ to ensure model's identifiability.

2.2. Estimation of *JS* circular regression parameters

There are two methods of estimating the parameters of the *JS* circular regression model, namely, the least squares (*LS*) see, (Ceylan & Parlakyıldız, 2017) and likelihood estimation method (*MLE*).

2.2.1. Least squares estimation

Let $(u_1, v_1), ..., (u_n, v_n)$ be a random circular sample of size n. The observational Equations (4) can be written in matrix form

$$\mathbf{V}^{(1)} = \mathbf{U}\boldsymbol{\lambda}^{(1)} + \boldsymbol{\varepsilon}^{(1)} \quad \text{and} \quad \mathbf{V}^{(2)} = \mathbf{U}\boldsymbol{\lambda}^{(2)} + \boldsymbol{\varepsilon}^{(2)} \quad (5)$$

Therefore, Equations (4) can be summarized as $V^{(1)} = (V_{11}; ::::; V_{1n})'$; $\mathbf{V}^{(2)} = (V_{21}, ..., V_{2n})'$, $\boldsymbol{\varepsilon}^{(1)} = (\varepsilon_{11}, ..., \varepsilon_{1n})$, $\boldsymbol{\varepsilon}^{(2)} = (\varepsilon_{21}, ..., \varepsilon_{2n})$. $\mathbf{U}_{n\times(2m+1)} =$

$$\begin{bmatrix} 1 & \cos u_1 & .. & \cos mu_1 & \sin u_1 & .. & \sin mu_1 \\ 1 & \cos u_2 & .. & \cos mu_1 & \sin u_2 & .. & \sin mu_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cos u_n & .. & \cos mu_n & \sin u_n & .. & \sin mu_n \end{bmatrix},$$

$$\boldsymbol{\lambda}^{(1)} = (A_0, A_1, ..., A_m, B_1, ..., B_m)'$$

and

$$\boldsymbol{\lambda}^{(2)} = (C_0, C_1, ..., C_m, D_1, ..., D_m)'. \quad (6)$$

The least squares estimates turn out to be

$$\hat{\boldsymbol{\lambda}}^{(1)} = \min \sum_{i=1}^{n} \left(\mathbf{V}_i^{(1)} - \mathbf{U}\boldsymbol{\lambda}^{(1)}\right)^2$$

and

$$\hat{\boldsymbol{\lambda}}^{(2)} = \min \sum_{i=1}^{n} \left(\mathbf{V}_i^{(2)} - \mathbf{U}\boldsymbol{\lambda}^{(2)}\right)^2. \quad (7)$$

These equations can be combined into the following single matrixes

$$\hat{\boldsymbol{\lambda}}^{(1)} = (\mathbf{U}'\mathbf{U})^{-1}\mathbf{U}'\mathbf{V}^{(1)} \quad \text{and} \quad \hat{\boldsymbol{\lambda}}^{(2)} = (\mathbf{U}'\mathbf{U})^{-1}\mathbf{U}'\mathbf{V}^{(2)} \quad (8)$$

2.2.2 Maximum likelihood estimation

An alternative estimation method JS circular regression models is the MLE method. For simplicity, we consider the case when m = 1. Hence, from Equations (4), we expand the error term and obtain

$$\varepsilon_j = (\cos v_j - A_0 - A_1 \cos u_j - B_1 \sin u_j) + i(\sin v_j - C_0 - C_1 \cos u_j - D_1 \sin u_j)$$

and

$$|\ \varepsilon_j\ |^2 = (\cos v_j - A_0 - A_1 \cos u_j - B_1 sin u_j)^2 + (\sin v_j - C_0 - C_1 \cos u_j - D_1 \sin u_j)^2.$$

Therefore, the log-likelihood function is given by

$$\log L(A_0, A_1, B_1, C_0, C_1, D_1, \sigma^2; u_j, v_j) = -n \log(\pi\sigma^2) - \frac{1}{\sigma^2}\sum_j (\cos v_j - A_0 - A_1 \cos u_j - B_1 \sin u_j)^2 - \frac{1}{\sigma^2}\sum_j (\sin v_j - C_0 - C_1 \cos u_j - D_1 \sin u_j)^2. \quad (9)$$

The function log L is then differentiated concerning each parameter and equated to zero. Hence, we obtain the following estimates of the parameters:

and

$$\hat{D}_1 = \frac{\sum_j (\sin v_j - \hat{C}_0 - \hat{C}_1 \cos u_j)}{\sum_j (\sin u_j)}.$$

Both *LS* and *MLE* methods should give similar estimates of the parameters *A0, A1, B1, C0, C1,* and *D1,* under the assumption that the error terms are normally distributed. The following section explains the effect of outliers on the *JS* circular regression model.

3. Outliers in the *JS* circular regression model

Outliers are a common problem in the statistical analysis. It is defined as observations that are very different from the other observations in a set of data. (Ibrahim *et al.*, 2013) investigated the robustness of the *JS* model by a simulation study and concluded that the *JS* model is sensitive for outliers exitance, and the presence of outliers has potentially several effects on *LS* estimation. Then (Ibrahim, 2013) proposed a *COV RATIO* statistic to define outliers in the y-vertical. This paper defines two types of outliers, outliers in **V** (vertical outliers), and outliers with respect to **U** (leverage points).

3.1 Effect of outliers on *LS* estimation of *JS* circular regression parameters

In order to illustrate the effect of outliers on *LS* estimation, we introduce the following two ways:

1. **Vertical outliers**: if $\mathbf{V}_{1j}$ and $\mathbf{V}_{2j}$ is replaced by $\mathbf{V}_{1j}^\star$ and $\mathbf{V}_{2j}^\star$, respectively, where, $\mathbf{V}_{1j}^\star = Z_1\mathbf{V}_{1j}$ and $\mathbf{V}_{2j}^\star = Z_2\mathbf{V}_{2j}$, where, $Z_{1,2} \in (-\infty, \infty)$, which implies $\mathbf{V}_{1j} = Z_1^{-1}\mathbf{V}_{1j}^\star$ and $\mathbf{V}_{2j} = Z_2^{-1}\mathbf{V}_{2j}^\star$, then the circular regression in (4) can be rewritten as follows: $Z_1^{-1}\mathbf{V}_{1j}^\star = \cos v_j$

$$= \sum_{k=0}^{m} (A_k \cos ku_j + B_k \sin ku_j) + \varepsilon_{1j}$$

and $Z_2^{-1}\mathbf{V}_{2j}^\star = \sin v_j$

$$= \sum_{k=0}^{m} (C_k \cos ku_j + D_k \sin ku_j) + \varepsilon_{2j},$$

consequently,

$$\hat{\lambda}^{(1)}(\mathbf{U},\mathbf{V}^{\star}_{1j}) = Z_1^{-1}\hat{\lambda}^{(1)}(\mathbf{U},\mathbf{V}_{1j}),$$

and

$$\hat{\lambda}^{(2)}(\mathbf{U},\mathbf{V}^{\star}_{2j}) = Z_2^{-1}\hat{\lambda}^{(2)}(\mathbf{U},\mathbf{V}_{2j}).$$

2. **Leverage points**: if $\mathbf{U}$ is replaced by $\mathbf{U}^{\star}$, where $\mathbf{U}^{\star} = Z\mathbf{U}$, then

$$\hat{\lambda}^{(1)}(\mathbf{U}^{\star},\mathbf{V}_{1j}) = Z^{-1}\hat{\lambda}^{(1)}(\mathbf{U},\mathbf{V}_{1j}),$$

and

$$\hat{\lambda}^{(2)}(\mathbf{U}^{\star},\mathbf{V}_{2j}) = Z^{-1}\hat{\lambda}^{(2)}(\mathbf{U},\mathbf{V}_{2j}).$$

However, it is clear that the circular parameters is very sensitive to the presence of outlying, and it will be affected dramatically according to the types of outliers. Therefore, before address the robustness issue in the *JS* circular regression model, it is necessary to discuss the finite sample breakdown point *(BDP)*.

3.2 Breakdown point

The *BDP* was first introduced in (Donoho & Huber, 2016). The *BDP* is defined in terms of the smallest fraction of observations which can be contaminated such that the estimator goes arbitrarily far from the estimator based on all the observations. Formally,

$$BDP = \inf\left\{\frac{h}{n} : \sup \|T(X) - T'(X)\| = \infty\right\},$$

where, T(*X*) is the estimator based on all the n observations, while T′(*X*) is the estimator when h out of n observations are contaminated arbitrarily.
The *BDP* of *LS* in the simple linear regression is 1 n, it means that one outlier can affect the *LS* estimation. For the *JS* circular regression model, we conclude the same result of *BDP* for circular *LS* estimation.
Similarly, the contamination in $(\mathbf{V},\mathbf{U})$ can make the *MLE* of circular parameters unbounded. The *BDP* define in terms of . Following the definition of *BDP*, the *BDP* of *MLE* is

$$\inf\left\{\frac{h}{n} : \sup \|\hat{\lambda} - \hat{\lambda}^{\star}\|\right\},$$

4. Circular least trimmed squares *(CLTS)* estimators

In linear regression, the *LS* estimator has *BDP* = 1=*n*. Thus, there was necessary to improve the robustness of the estimator because a single outlier has the power to change the value of the *LS* estimate arbitrarily. In terms of the *BDP*, the issue rest with the introduction of robust estimators such as the least median of squares estimator proposed by (Rousseeuw, 1984) and M-estimator see, (Huber, 2011). Thus, a more robust estimator with a higher BDP was needed.
In follow-up of such a robust estimator, (Rousseeuw, 1985) introduced *LTS* estimator which has *BDP* = ([*n*=2+1]=*n*) for simple linear regression; this section extends the *LTS* estimator idea to *JS* circular regression case, and proposes the circular least trimmed squares estimators (*CLTS*). In the linear regression, the prediction accuracy of an estimator computed by the difference between the observed values and the predicted values, smaller values of the squares of the distance between the observed and predicted values, is the best fitting. Considering the linear regression case, this discrepancy in the *JS* circular regression can be measured by taking the distance between the observed *V* and predicting the responses *V*. The *JS* circular residual is defined as $\varepsilon^{(1)}_{(i)} = V^{(1)}-U^{(1)}$ and $\varepsilon^{(2)}_{(i)} = V^{(2)}-U^{(2)}$, the lowest value of this residual is the best fitting
Thus, by ordering the squared residuals ε_1^2 and ε_2^2 as

$$\varepsilon^2_{1(1)}, \varepsilon^2_{1(2)}, \ldots, \varepsilon^2_{1(n)}$$

and

$$\varepsilon^2_{2(1)}, \varepsilon^2_{2(2)}, \ldots, \varepsilon^2_{2(n)},$$

the $CLTS$ define as the value of the parameters $\lambda^{(1)}$ and $\lambda^{(2)}$ which minimizes

$$\sum_{i=1}^{h}(\varepsilon^{(1)})_{(i)} \quad \text{and} \quad \sum_{i=1}^{h}(\varepsilon^{(2)})_{(i)},$$

where, [n/2] + 1>*h*> *n*. This estimator is based on h observations out of n. Typically, the valueof h is taken to be greater than [n/2] because a lower value of h means that more than half of the observations are contaminated, which does not create considerable meaning. In this paper we suggest that h = n+4/2 analogous to (Jha & Biswas, 2017). However, the *CLTS* is the value of the parameter, which fits h observations out of the total n in the best way. This is gained in the selfsame as the maximum *LTS* likelihood estimator was obtained as the robust answer from the classical maximum likelihood estimator (*MLE*). See, e.g., (Vandev & Neykov, 1993), (Bednarski *et al*., 1993), (Vandev & Neykov, 1998) and (Cuesta *et al*., 2008) for the descriptions, rationale and applicability of the maximum *LTS* likelihood estimator. For circular data, the maximum *LTS* likelihood estimator will be the same as the *CLTS*. However, the residual is the natural technique of demonstrating the goodness of fit in circular data. Hence this paper focuses on the *CLTS* instead of the circular maximum *LTS* likelihood estimator.

5. Simulation study

A simulation study was achieved to examine the proposed *CLTS* estimates performance and compare it with classical *LS* of *JS* circular regression models. For simplicity, we consider the case when m = 1, so the six coefficients are to be estimated; A_0, A_1, B_1, C_0, C_1, and D_1. Specifically, we consider the set of uncorrelated random errors (ε_1; ε_2) from the bivariate Normal distribution with mean vector **0** and variances (a_1;a_2) to be (0.03,0.03). For simplicity and illustrative purposes, the true values of A_0 and C_0 being zero, while A_1, B_1, C_0, and D_1 are obtained by using the standard additive trigonometric polynomial equations *cos(a+ u)* and *sin(a + u)*: For example, when a = 2, $cos(2 + u)$ = -0:0416 $cos\ u$ -0:9093 $sin\ u$ and $sin(2+u)$ = 0:9093 $cos\ u$ -0:04161 $sin\ u$: Then by comparing with equation (4), the true values of A_1, B_1, C_1, and D_1 are -0.04161, -0.09093, 0.09093, and -0.04161, respectively. Similarly, we can also get different sets of true values by choosing different values of a. Here, we consider the values of a = -6 and 2. We then introduce outliers into the data such that the percentages of contamination used is $c\%$= 10%, 20%, and 30% from the sample size n =20, 50, 100. The complete steps of the simulation are described below:

Step 1. Generate a fixed variable **U** of size n from VM(; 2).

Step 2. Generate ε1 and ε2 of size n from N $(\binom{0}{0};,\begin{pmatrix}0{:}03 & 0\\ 0 & 0{:}03\end{pmatrix})$. For a fixed a, obtain the true values of A_1, B_1, C_1 and D_1, then calculate V_{1j} and V_{2j}.

Step 3. Obtain the variable vj = arctan V_{2j} V_{1j}.

Step 4. Fit the generated circular data to the *JS* circular regression model to give the *LS* and *CLTS* parameter estimates $\hat{A}_0$, $\hat{A}_1$, $\hat{B}_1$, $\hat{C}_0$, $\hat{C}_1$, and $\hat{D}_1$. The process is carried out 1000 times for each combination of sample size n and different values of a. To investigate the robustness of the estimators against vertical outliers and leverage points, the following scenarios were considered:

1. No contamination.

2. Vertical outliers (outliers in the **V** only).

3. Leverage points (outliers in some **U** only).

For the $c\%$ vertical outliers scenario, we replace the first $cxn/100$ observations v with the newly generated values v such that the errors ε1 and ε2 are now generated from $N(\binom{0}{0}\begin{pmatrix}0{:}5 & 0\\ 0 & 0{:}5\end{pmatrix})$ Then, the generated contaminated circular data are fitted using *LS* and *CLTS*. For the $c\%$ leverage point scenario, we replace the first $cxn/100$ observations u by the newly generated values u_d $VM(2,6)$ instead of the original generated data from $VM(; 2)$. For each parameter estimates, the estimated mean, bias, standard error (SE) and root mean squared error (RMSE) are calculated using the following formulas:

- Mean of the estimates is given by $\bar{\lambda}_i = \frac{\sum_{j=1}^{1000}\hat{\lambda}_{i,j}}{1000}, i, 1, 2, ..., 6.$
- SE of the estimates is given by $SE(\hat{\lambda}_i) = \sqrt{\frac{\sum_{j=1}^{1000}(\hat{\lambda}_{i,j}-\bar{\lambda}_i)^2}{1000}}.$
- The bias of the estimates is given by $bias(\hat{\lambda}_i) = \bar{\lambda}_i - \lambda_i.$
- $RMSE$ of the estimates is given by $RMSE(\hat{\lambda}_i) = \sqrt{\frac{\sum_{j=1}^{1000}(\hat{\lambda}_{i,j}-\lambda_i)^2}{1000}}.$

The simulations were performed by the statistical software *R*. Two *R*-packages are used, *'CircStats'* is used to generate circular variables, the function *'rvm0'* is used to generate a set of circular random variables from the von Mises distribution VM(μ,κ). *'MASS'* is used to run the proposed estimator *CLTS*, the function *'ltsreg'* is used to fit a linear regression model; this function is adopted to fit a *JS* circular regression model.

The results are tabulated in Tables 1 to 6 for each value of a considered. We find that the power performance of the *CLTS* estimator. As expected, For outlier-free data set, both *LS* and *CLTS* estimated mean for all parameter estimates are consistently close to the actual values. When the percentage of contamination increases from 10% to 30%, the *LS* value of the bias increases. Otherwise, *CLTS* shows relatively small bias results. The *SE* for all parameters *LS* estimates are generally small for uncontaminated data but get larger as the percentages of contamination increase. The *RMSE* of each *LS* parameter estimates increases when the percentages of contamination increase. On the other hand, we find that the performance of the *CLTS* estimator is an increasing function of n and becomes similar when the percentage of contamination

Table 1. Simulation results of *LS* and *CLTS* estimators for uncontaminated data when a=-6

			LS				*CLTS*			
n	Estimates	True value	Mean	SE	Bias	RMSE	Mean	SE	Bias	RMSE
20	$\hat{A}_0$	0.0000	9.1e-05	1.3e-02	9.1e-05	2.9e-03	-2.0e-03	3.7e-02	-2.0e-03	6.4e-02
	$\hat{A}_1$	0.9602	0.9598	0.0155	-0.0002	0.0089	0.9578	0.0381	-0.0022	0.0724
	$\hat{B}_1$	-0.2794	-2.7e-01	9.4e-03	6.4e-05	2.0e-03	-2.7e-01	1.5e-02	7.5e-04	2.3e-02
	$\hat{C}_0$	0.0000	0.0002	0.0102	0.0002	0.0085	-0.0006	0.0326	-0.0006	0.0217
	$\hat{C}_1$	0.2794	0.2797	0.0139	0.0003	0.0103	0.2789	0.0388	-0.0004	0.0139
	$\hat{D}_1$	0.9602	0.9595	0.0086	-0.0006	0.0208	0.9599	0.0170	-0.0001	0.0057
50	$\hat{A}_0$	0.0000	-6.2e-05	8.0e-03	-6.2e-05	1.9e-03	-1.1e-03	2.4e-02	-1.1e-03	3.5e-02
	$\hat{A}_1$	0.9602	0.9597	0.0089	-0.0004	0.0146	0.9591	0.0244	-0.0010	0.0327
	$\hat{B}_1$	-0.2794	-0.2792	0.0062	0.0001	0.0042	-0.2785	0.0114	0.0008	0.0263
	$\hat{C}_0$	0.0000	-0.0002	0.0054	-0.0002	0.0072	0.0005	0.0119	0.0005	0.0170
	$\hat{C}_1$	0.2794	0.2789	0.0078	-0.0004	0.0158	0.2799	0.0169	0.0005	0.0177
	$\hat{D}_1$	0.9602	0.9597	0.0047	-0.0003	0.0120	0.9605	0.0089	0.0003	0.0114
100	$\hat{A}_0$	0.0000	-0.0001	0.0051	-0.0001	0.0062	-0.0013	0.0183	-0.0013	0.0440
	$\hat{A}_1$	0.9602	0.9595	0.0056	-0.0005	0.0188	0.9586	0.0184	-0.0014	0.0471
	$\hat{B}_1$	-0.2794	-0.2793	0.0041	0.0000	0.0029	-0.2791	0.0093	0.0002	0.0090
	$\hat{C}_0$	0.0000	-1.6e-04	3.6e-03	-1.6e-04	5.3e-03	-8.7e-05	5.4e-03	-8.7e-05	2.7e-03
	$\hat{C}_1$	0.2794	0.2789	0.0053	-0.0004	0.0150	0.2792	0.0110	-0.0001	0.0040
	$\hat{D}_1$	0.9602	0.9597	3.2e-03	-4.3e-04	0.0139	0.9602	5.1e-03	6.7e-05	2.1e-03

Table 2. Simulation results of *LS* and *CLTS* estimators for data with verticals when a=-6

				LS				CLTS			
	n	Estimates	True value	Mean	SE	Bias	RMSE	Mean	SE	Bias	RMSE
10% verticals	20	$\hat{A}_0$	0.0000	-0.0225	0.1101	-0.0225	0.7136	-0.0011	0.0354	-0.0011	0.0356
		$\hat{A}_1$	0.9602	0.9157	0.1296	-0.0444	1.4045	0.9590	0.0357	-0.0011	0.0356
		$\hat{B}_1$	-0.2794	-0.2672	0.0621	0.0121	0.3847	-0.2783	0.0163	0.0010	0.0339
		$\hat{C}_0$	0.0000	-0.0110	0.1534	-0.0110	0.3508	-0.0014	0.0367	-0.0014	0.0461
		$\hat{C}_1$	0.2794	0.2594	0.1697	-0.0199	0.6312	0.2776	0.0432	-0.0017	0.0544
		$\hat{D}_1$	0.9602	0.9136	0.0831	-0.0465	1.4708	0.9612	0.0183	0.0011	0.0355
10% verticals	50	$\hat{A}_0$	0.0000	-0.0267	0.0671	-0.0267	0.8445	-0.0017	0.0226	-0.0017	0.0559
		$\hat{A}_1$	0.9602	0.9120	0.0803	-0.0481	1.5226	0.9581	0.0230	-0.0019	0.0627
		$\hat{B}_1$	-0.2794	-0.2647	0.0373	0.0147	0.4652	-0.2792	0.0110	0.0001	0.0061
		$\hat{C}_0$	0.0000	-0.0083	0.0735	-0.0083	0.2651	0.0004	0.0100	0.0004	0.0157
		$\hat{C}_1$	0.2794	0.2648	0.0841	-0.0146	0.4618	0.2805	0.0153	0.0011	0.0359
		$\hat{D}_1$	0.9602	0.9139	0.0511	-0.0462	1.4624	0.9601	8.8e-03	-8.9e-06	2.8e-04
10% verticals	100	$\hat{A}_0$	0.0000	-0.0237	0.0471	-0.0237	0.7495	-0.0021	0.0170	-0.0021	0.0679
		$\hat{A}_1$	0.9602	0.9142	0.0561	-0.0458	1.4508	0.9579	0.0169	-0.0022	0.0703
		$\hat{B}_1$	-0.2794	-0.2661	0.0265	0.0132	0.4197	-0.2789	0.0092	0.0004	0.0134
		$\hat{C}_0$	0.0000	-0.0081	0.0481	-0.0081	0.2565	0.0003	0.0058	0.0003	0.0116
		$\hat{C}_1$	0.2794	0.2652	0.0557	-0.0141	0.4486	0.2797	0.0108	0.0003	0.0105
		$\hat{D}_1$	0.9602	0.9111	0.0361	-0.0490	1.5510	0.9600	0.0049	-0.0001	0.0041
20% verticals	20	$\hat{A}_0$	0.0000	-0.0444	0.1488	-0.0444	1.4060	0.0028	0.0326	0.0028	0.0894
		$\hat{A}_1$	0.9602	0.8730	0.1729	-0.0871	2.7555	0.9630	0.0334	0.0028	0.0914
		$\hat{B}_1$	-0.2794	-0.2546	0.0850	0.0247	0.7829	-0.2794	0.0153	-7.3e-05	2.33e-03
		$\hat{C}_0$	0.0000	-0.0216	0.2055	-0.0216	0.6846	0.0001	0.0345	0.0001	0.0057
		$\hat{C}_1$	0.2794	0.2434	0.2300	-0.0360	1.1385	0.2798	0.0403	0.0004	0.0149
		$\hat{D}_1$	0.9602	0.8619	0.1209	-0.0981	3.1046	0.9587	0.0193	-0.0013	0.0433
20% verticals	50	$\hat{A}_0$	0.0000	-0.0542	0.0907	-0.0542	1.7159	-0.0001	0.0238	-0.0001	0.0033
		$\hat{A}_1$	0.9602	0.8622	0.1117	-0.0979	3.0974	0.9600	0.0242	-0.0001	0.0038
		$\hat{B}_1$	-0.2794	-0.2525	0.0509	0.0268	0.8489	-0.2792	0.0114	0.0001	0.0057
		$\hat{C}_0$	0.0000	-0.0174	0.1046	-0.0174	0.5503	0.0000	0.0100	0.0000	0.0011
		$\hat{C}_1$	0.2794	0.2480	0.1184	-0.0313	0.9925	0.2795	0.0159	0.0001	0.0036
		$\hat{D}_1$	0.9602	0.8605	0.0698	-0.0995	3.1495	0.9598	0.0083	-0.0003	0.0102
20% verticals	100	$\hat{A}_0$	0.0000	-0.0490	0.0655	-0.0490	1.5515	-0.0011	0.0172	-0.0011	0.0351
		$\hat{A}_1$	0.9602	0.8679	0.0782	-0.0921	2.9148	0.9591	0.0173	-0.0010	0.0327
		$\hat{B}_1$	-0.2794	-0.2499	0.0362	0.0294	0.9307	-0.2784	0.0090	0.0009	0.0312
		$\hat{C}_0$	0.0000	-0.0142	0.0688	-0.0142	0.4513	0.0001	0.0058	0.0001	0.0043
		$\hat{C}_1$	0.2794	0.2518	0.0771	-0.0275	0.8713	0.2795	0.0108	9.4e-05	3.0e-03
		$\hat{D}_1$	0.9602	0.8650	0.0487	-0.0951	3.0095	0.9598	0.0050	-0.0003	0.0100
30% verticals	20	$\hat{A}_0$	0.0000	-0.0883	0.1912	-0.0883	2.7928	-0.0003	0.0366	-0.0003	0.0095
		$\hat{A}_1$	0.9602	0.8024	0.2225	-0.1577	4.9888	0.9597	0.0374	-0.0004	0.0146
		$\hat{B}_1$	-0.2794	-0.2376	0.1057	0.0417	1.3206	-0.2789	0.0161	0.0004	0.0140
		$\hat{C}_0$	0.0000	-0.0171	0.2425	-0.0171	0.5435	0.0012	0.0384	0.0012	0.0397
		$\hat{C}_1$	0.2794	0.2439	0.2725	-0.0354	1.1223	0.2809	0.0457	0.0014	0.0470
		$\hat{D}_1$	0.9602	0.8158	0.1540	-0.1443	4.5652	0.9596	0.0197	-0.0005	0.0162
30% verticals	50	$\hat{A}_0$	0.0000	-0.0708	0.1078	-0.0708	2.2389	-0.0010	0.0223	-0.0010	0.0343
		$\hat{A}_1$	0.9602	0.8237	0.1302	-0.1364	4.3145	0.9589	0.0227	-0.0012	0.0396
		$\hat{B}_1$	-0.2794	-0.2389	0.0629	0.0404	1.2798	-0.2791	0.0112	0.0002	0.0092
		$\hat{C}_0$	0.0000	-0.0267	0.1265	-0.0267	0.84534	0.0006	0.0106	0.0006	0.0194
		$\hat{C}_1$	0.2794	0.2349	0.1463	-0.0444	1.4067	0.2803	0.0161	0.0009	0.0295
		$\hat{D}_1$	0.9602	0.8173	0.0875	-0.1428	4.5162	0.9603	0.0083	0.0002	0.0066
30% verticals	100	$\hat{A}_0$	0.0000	-0.0777	0.0819	-0.0777	2.4579	-0.0023	0.0172	-0.0023	0.0732
		$\hat{A}_1$	0.9602	0.8143	0.0981	-0.1458	4.6121	0.9578	0.0174	-0.0023	0.0746
		$\hat{B}_1$	-0.2794	-0.2363	0.0456	0.0430	1.3619	-0.2785	0.0088	0.0008	0.0261
		$\hat{C}_0$	0.0000	-0.0256	0.0828	-0.0256	0.8123	-0.0002	0.0057	-0.0002	0.0063
		$\hat{C}_1$	0.2794	0.2356	0.0971	-0.0437	1.3829	0.2796	0.0107	0.0002	0.0069
		$\hat{D}_1$	0.9602	0.8161	0.0600	-0.1440	4.5542	0.9596	0.0052	-0.0004	0.0157

Table 3. Simulation results of *LS* and *CLTS* estimators for data with leverage point when a=-6

				LS				CLTS			
	n	Estimates	True value	Mean	SE	Bias	RMSE	Mean	SE	Bias	RMSE
10% leverage points	20	$\hat{A}_0$	0.0000	-0.4745	0.1993	-0.4745	15.0068	-0.0035	0.0345	-0.0035	0.1128
		$\hat{A}_1$	0.9602	0.3797	0.2308	-0.5803	18.3530	0.9565	0.0354	-0.0036	0.1148
		$\hat{B}_1$	-0.2794	-0.1841	0.1516	0.0953	3.0138	-0.2788	0.0149	0.0005	0.0185
		$\hat{C}_0$	0.0000	-0.0561	0.2551	-0.0561	1.7758	-0.0006	0.0408	-0.0006	0.0203
		$\hat{C}_1$	0.2794	0.1831	0.2948	-0.0962	3.0436	0.2789	0.0458	-0.0004	0.0148
		$\hat{D}_1$	0.8151	0.1309	-0.1450	4.5874	0.9600	0.0191	-9.2e-05	2.9e-03	
10% leverage points	50	$\hat{A}_0$	0.0000	-0.4495	0.1170	-0.4495	14.2150	-0.0012	0.0237	-0.0012	0.0409
		$\hat{A}_1$	0.9602	0.4041	0.1363	-0.5560	17.5835	0.9588	0.0240	-0.0012	0.0408
		$\hat{B}_1$	-0.2794	-0.1703	0.0851	0.1090	3.4478	-0.2786	0.0115	0.0007	0.0227
		$\hat{C}_0$	0.0000	-0.0771	0.1365	-0.0771	2.4405	0.0002	0.0104	0.0002	0.0087
		$\hat{C}_1$	0.2794	0.1537	0.1601	-0.1256	3.9740	0.2799	0.0162	0.0005	0.0182
		$\hat{D}_1$	0.9602	0.7960	0.0762	-0.1640	5.1888	0.9603	0.0087	0.0001	0.0044
10% leverage points	100	$\hat{A}_0$	0.0000	-0.4397	0.0778	-0.4397	13.9064	-0.0015	0.0170	-0.0015	0.0500
		$\hat{A}_1$	0.9602	0.4147	0.0921	-0.5453	17.2463	0.9585	0.0173	-0.0015	0.0499
		$\hat{B}_1$	-0.2794	-0.1687	0.0581	0.1106	3.4991	-0.2786	0.0090	0.0007	0.0238
		$\hat{C}_0$	0.0000	-0.0798	0.0913	-0.0798	2.5263	7.4e-05	5.2e-03	7.4e-05	2.3e-03
		$\hat{C}_1$	0.2794	0.1491	0.1062	-0.1302	4.1185	0.2794	0.0106	-3.8e-06	1.2e-04
		$\hat{D}_1$	0.9602	0.7898	0.0537	-0.1703	5.3857	0.9602	5.3e-03	6.8e-05	2.1e-03
20% leverage points	20	$\hat{A}_0$	0.0000	-0.5940	0.1596	-0.5940	18.7863	-0.0021	0.0343	-0.0021	0.0673
		$\hat{A}_1$	0.9602	0.2233	0.1784	-0.7368	23.2999	0.9580	0.0353	-0.0021	0.0674
		$\hat{B}_1$	-0.2794	-0.1150	0.1838	0.1643	5.1983	-0.2783	0.0156	0.0010	0.0322
		$\hat{C}_0$	0.0000	-0.1104	0.1989	-0.1104	3.4935	-0.0011	0.0569	-0.0011	0.0348
		$\hat{C}_1$	0.2794	0.0835	0.2256	-0.1959	6.1952	0.2777	0.0617	-0.0016	0.0533
		$\hat{D}_1$	0.9602	0.6458	0.1682	-0.3143	9.9412	0.9601	0.0204	-1.3e-05	4.3e-04
20% leverage points	50	$\hat{A}_0$	0.0000	-0.5734	0.0943	-0.5734	18.1332	-0.0009	0.0234	-0.0009	0.0315
		$\hat{A}_1$	0.9602	0.2420	0.1070	-0.7181	22.7102	0.9589	0.0238	-0.0011	0.0370
		$\hat{B}_1$	-0.2794	-0.0971	0.1042	0.1822	5.7638	-0.2791	0.0113	0.0003	0.0095
		$\hat{C}_0$	0.0000	-0.1192	0.1196	-0.1192	3.7698	0.0002	0.0115	0.0002	0.0086
		$\hat{C}_1$	0.2794	0.0698	0.1349	-0.2095	6.6279	0.2798	0.0169	0.0004	0.0137
		$\hat{D}_1$	0.9602	0.6222	0.1003	-0.3379	10.6874	0.9600	0.0085	-0.0001	0.0053
20% leverage points	100	$\hat{A}_0$	0.0000	-0.5669	0.0675	-0.5669	17.9282	-0.0009	0.0157	-0.0009	0.0297
		$\hat{A}_1$	0.9602	0.2496	0.0757	-0.7104	22.4672	0.9590	0.0160	-0.0011	0.0356
		$\hat{B}_1$	-0.2794	-0.0947	0.0726	0.1846	5.8391	-0.2793	0.0087	0.0001	0.0034
		$\hat{C}_0$	0.0000	-0.1181	0.0799	-0.1181	3.7369	0.0001	0.0053	0.0001	0.0033
		$\hat{C}_1$	0.2794	0.0699	0.0897	-0.2094	6.6244	0.2799	0.0099	0.0005	0.0184
		$\hat{D}_1$	0.9602	0.6203	0.0690	-0.3398	10.7467	0.9598	0.0053	-0.0003	0.0095
30% leverage points	20	$\hat{A}_0$	0.0000	-0.6383	0.1317	-0.6383	20.1862	-0.0016	0.0307	-0.0016	0.0510
		$\hat{A}_1$	0.9602	0.1566	0.1419	-0.8035	25.4104	0.9586	0.0324	-0.0014	0.0466
		$\hat{B}_1$	-0.2794	-0.0463	0.1904	0.2331	7.3713	-0.2788	0.0147	0.0006	0.0192
		$\hat{C}_0$	0.0000	-0.1394	0.1666	-0.1394	4.4084	0.0011	0.0509	0.0011	0.0363
		$\hat{C}_1$	0.2794	0.0152	0.1775	-0.2641	8.3546	0.2800	0.0554	0.0006	0.0204
		$\hat{D}_1$	0.9602	0.4525	0.1872	-0.5075	16.0511	0.9607	0.0229	0.0005	0.0167
30% leverage points	50	$\hat{A}_0$	0.0000	-0.6200	0.0793	-0.6200	19.6087	-0.0018	0.0201	-0.0018	0.0573
		$\hat{A}_1$	0.9602	0.1695	0.0860	-0.7906	25.0020	0.9580	0.0207	-0.0021	0.0664
		$\hat{B}_1$	-0.2794	-0.0302	0.1056	0.2491	7.8781	-0.2790	0.0103	0.0003	0.0101
		$\hat{C}_0$	0.0000	-0.1479	0.0976	-0.1479	4.6794	-6.2e-06	0.0114	-6.2e-06	1.9e-04
		$\hat{C}_1$	0.2794	1.6e-03	0.1076	-0.2777	8.7823	0.2794	0.0167	8.0e-06	2.5e-04
		$\hat{D}_1$	0.9602	0.4493	0.1148	-0.5108	16.1550	0.9601	0.0084	0.0000	0.0005
30% leverage points	100	$\hat{A}_0$	0.0000	-0.6226	0.0563	-0.6226	19.6901	-0.0008	0.0140	-0.0008	0.0271
		$\hat{A}_1$	0.9602	0.1668	0.0607	-0.7933	25.0882	0.9590	0.0145	-0.0010	0.0345
		$\hat{B}_1$	-0.2794	-0.0320	0.0763	0.2473	7.8229	-0.2792	0.0082	0.0001	0.0059
		$\hat{C}_0$	0.0000	-0.1466	0.0687	-0.1466	4.6368	0.0001	0.0060	0.0001	0.0045
		$\hat{C}_1$	0.2794	0.0024	0.0742	-0.2770	8.7595	0.2797	0.0105	0.0003	0.0101
		$\hat{D}_1$	0.9602	0.4487	0.0776	-0.5114	16.1729	0.9600	0.0051	-0.0001	0.0034

Table 4. Simulation results of *LS* and *CLTS* estimators for uncontaminated data when a=2

			LS				CLTS			
n	Estimates	True value	Mean	SE	Bias	RMSE	Mean	SE	Bias	RMSE
20	$\hat{A}_0$	0.0000	-0.0001	0.0117	-0.0001	0.0044	0.0001	0.0291	0.0001	0.0055
	$\hat{A}_1$	-0.4161	-0.4160	0.0154	9.0e-05	2.8e-03	-0.4157	0.0343	4.3e-04	0.0137
	$\hat{B}_1$	0.9093	-0.9083	0.0086	0.0009	0.0292	-0.9087	0.0178	0.0005	0.0180
	$\hat{C}_0$	0.0000	0.0002	0.0142	0.0002	0.0063	-0.0002	0.0344	0.0002	0.0078
	$\hat{C}_1$	-0.9093	0.9093	0.0160	7.0e-05	2.2e-03	0.9093	0.0350	2.2e-05	7.1e-04
	$\hat{D}_1$	-0.4161	-0.4158	0.0091	0.0002	0.0084	-0.4148	0.0176	0.0012	0.0408
50	$\hat{A}_0$	0.0000	-5.9e-06	5.6e-03	-5.9e-06	1.8e-04	-4.2e-04	0.0140	-4.2e-04	0.0132
	$\hat{A}_1$	-0.4161	-0.4160	0.0077	0.0001	0.0040	-0.4167	0.0180	-0.0005	0.0186
	$\hat{B}_1$	0.9093	-0.9089	0.0049	0.0003	0.0102	-0.9094	0.0095	-0.0001	0.0048
	$\hat{C}_0$	0.0000	0.0000	0.0075	0.0000	0.0026	0.0002	0.0220	0.0002	0.0074
	$\hat{C}_1$	-0.9093	0.9088	0.0085	-0.0004	0.0134	0.9095	0.0222	0.0002	0.0088
	$\hat{D}_1$	-0.4161	-0.4161	5.4e-03	-3.1e-05	1.0e-03	-0.4158	0.0119	2.9e-04	9.3e-03
100	$\hat{A}_0$	0.0000	-4.0e-06	3.9e-03	-4.0e-06	1.2e-04	-3.6e-04	7.1e-03	-3.6e-04	0.0115
	$\hat{A}_1$	-0.4161	-0.4159	0.0055	0.0001	0.0058	-0.4165	0.0112	-0.0003	0.0113
	$\hat{B}_1$	0.9093	-0.9088	0.0033	0.0004	0.0137	-0.9094	0.0062	-0.0001	0.0055
	$\hat{C}_0$	0.0000	-8.9e-05	5.0e-03	-8.9e-05	2.8e-03	-9.6e-06	0.0161	-9.6e-06	3.0e-04
	$\hat{C}_1$	-0.9093	0.9088	0.0056	-0.0004	0.0155	0.9095	0.0163	0.0002	0.0083
	$\hat{D}_1$	-0.4161	-0.4159	0.0041	0.0002	0.0066	-0.4154	0.0092	0.0007	0.0231

Table 5. Simulation results of *LS* and *CLTS* estimators for data with verticals when a=2

				LS				$CLTS$			
	n	Estimates	True value	Mean	SE	Bias	RMSE	Mean	SE	Bias	RMSE
10% verticals	20	$\hat{A}_0$	0.0000	0.0119	0.1380	0.0119	0.3790	-0.0004	0.0433	-0.0004	0.0152
		$\hat{A}_1$	-0.4161	-0.3960	0.1551	0.0200	0.6351	-0.4160	0.0476	9.8e-05	3.1e-03
		$\hat{B}_1$	0.9093	-0.8630	0.0852	0.0462	1.4625	-0.9098	0.0183	-0.0005	0.0176
		$\hat{C}_0$	0.0000	-0.0246	0.1134	-0.0246	0.7808	0.0011	0.0334	0.0011	0.0355
		$\hat{C}_1$	-0.9093	0.8638	0.1325	-0.0454	1.4367	0.9104	0.0348	0.0011	0.0378
		$\hat{D}_1$	-0.4161	-0.3921	0.0667	0.0240	0.7600	-0.4155	0.0159	0.0005	0.0177
10% verticals	50	$\hat{A}_0$	0.0000	0.0144	0.0774	0.0144	0.4583	-0.0004	0.0119	-0.0004	0.0128
		$\hat{A}_1$	-0.4161	-0.3921	0.0895	0.0240	0.7597	-0.4162	0.0159	-0.0001	0.0047
		$\hat{B}_1$	0.9093	-0.8633	0.0509	0.0459	1.4520	-0.9093	9.3e-03	-4.7e-05	1.4e-03
		$\hat{C}_0$	0.0000	-0.0269	0.0731	-0.0269	0.8509	-0.0003	0.0207	-0.0003	0.0107
		$\hat{C}_1$	-0.9093	0.8606	0.0873	-0.0486	1.5380	0.9090	0.0210	-0.0002	0.0077
		$\hat{D}_1$	-0.4161	-0.3956	0.0390	0.0204	0.6468	-0.4154	0.0117	0.0007	0.0227
10% verticals	100	$\hat{A}_0$	0.0000	0.0103	0.0471	0.0103	0.3287	7.8e-06	6.4e-03	7.8e-06	2.4e-04
		$\hat{A}_1$	-0.4161	-0.3970	0.0534	0.0191	0.6050	-0.4164	0.0104	-0.0003	0.0105
		$\hat{B}_1$	0.9093	-0.8657	0.0352	0.0435	1.3781	-0.9090	0.0063	0.0002	0.0078
		$\hat{C}_0$	0.0000	-0.0206	0.0439	-0.0206	0.6525	-0.0003	0.0151	-0.0003	0.0100
		$\hat{C}_1$	-0.9093	0.8677	0.0532	-0.0415	1.3132	0.9090	0.01568	-0.0002	0.0086
		$\hat{D}_1$	-0.4161	-0.3966	0.0266	0.0194	0.6154	-0.4157	0.0089	0.0003	0.0112
20% verticals	20	$\hat{A}_0$	0.0000	0.0243	0.2109	0.02435	0.7700	-0.0003	0.0442	-0.0003	0.0109
		$\hat{A}_1$	-0.4161	-0.3725	0.2402	0.0435	1.3773	-0.4176	0.0488	-0.0015	0.0477
		$\hat{B}_1$	0.9093	-0.8200	0.1162	0.0892	2.8235	-0.9085	0.0214	0.0007	0.0234
		$\hat{C}_0$	0.0000	-0.0594	0.1641	-0.0594	1.8793	-0.0004	0.0381	-0.0004	0.0134
		$\hat{C}_1$	-0.9093	0.8080	0.1917	-0.1012	3.2018	0.9084	0.0375	-0.0008	0.0271
		$\hat{D}_1$	-0.4161	-0.3757	0.0888	0.0404	1.2788	-0.4161	0.0197	-5.2e-05	1.6e-03
20% verticals	50	$\hat{A}_0$	0.0000	0.0299	0.1057	0.0299	0.9479	0.0003	0.0118	0.0003	0.0121
		$\hat{A}_1$	-0.4161	-0.3690	0.1217	0.0471	1.4901	-0.4158	0.0167	0.0003	0.0101
		$\hat{B}_1$	0.9093	-0.8162	0.0710	0.0930	2.9410	-0.9087	0.0094	0.0005	0.0159
		$\hat{C}_0$	0.0000	-0.0505	0.1014	-0.0505	1.5981	-0.0012	0.0220	-0.0012	0.0406
		$\hat{C}_1$	-0.9093	0.8186	0.1196	-0.0906	2.8663	0.9079	0.0222	-0.0013	0.0415
		$\hat{D}_1$	-0.4161	-0.3728	0.0551	0.0432	1.3686	-0.4157	0.0119	0.0004	0.0129
20% verticals	100	$\hat{A}_0$	0.0000	0.0185	0.0696	0.0185	0.5877	0.0002	0.0065	0.0002	0.0077
		$\hat{A}_1$	-0.4161	-0.3799	0.0791	0.0362	1.1454	-0.4156	0.0110	0.0005	0.0169
		$\hat{B}_1$	0.9093	-0.8158	0.0493	0.0934	2.9543	-0.9092	6.2e-03	1.7e-05	5.5e-04
		$\hat{C}_0$	0.0000	-0.0489	0.0659	-0.0489	1.5494	-0.0009	0.0151	-0.0009	0.0303
		$\hat{C}_1$	-0.9093	0.8179	0.0781	-0.0913	2.8900	0.9082	0.0153	-0.0009	0.0315
		$\hat{D}_1$	-0.4161	-0.3724	0.0399	0.0437	1.3823	-0.4154	0.0088	0.0006	0.0211
30% verticals	20	$\hat{A}_0$	0.0000	0.0355	0.2391	0.0355	1.1242	-0.0007	0.0452	-0.0007	0.0222
		$\hat{A}_1$	-0.4161	-0.3512	0.2676	0.0649	2.0536	-0.4172	0.0497	-0.0011	0.0354
		$\hat{B}_1$	0.9093	-0.7635	0.1481	0.1457	4.6075	-0.9081	0.0200	0.0011	0.0362
		$\hat{C}_0$	0.0000	-0.0825	0.1961	-0.0825	2.6109	-0.0010	0.0347	-0.0010	0.0335
		$\hat{C}_1$	-0.9093	0.7632	0.2261	-0.1460	4.6177	0.9081	0.0364	-0.0011	0.0349
		$\hat{D}_1$	-0.4161	-0.3461	0.1121	0.0699	2.2123	-0.4159	0.0159	0.0002	0.0077
30% verticals	50	$\hat{A}_0$	0.0000	0.0370	0.1221	0.0370	1.1722	-0.0005	0.0121	-0.0005	0.0165
		$\hat{A}_1$	-0.4161	-0.3523	0.1365	0.0638	2.0184	-0.4170	0.0169	-0.0008	0.0275
		$\hat{B}_1$	0.9093	-0.7719	0.0852	0.1372	4.3417	-0.9091	0.0093	0.0001	0.0047
		$\hat{C}_0$	0.0000	-0.0723	0.1140	-0.0723	2.2888	-0.0012	0.0207	-0.0012	0.0384
		$\hat{C}_1$	-0.9093	0.7740	0.1361	-0.1352	4.2756	0.9079	0.0211	-0.0013	0.0440
		$\hat{D}_1$	-0.4161	-0.3548	0.0641	0.0613	1.9396	-0.4159	0.0116	0.0002	0.0069
30% verticals	100	$\hat{A}_0$	0.0000	0.0343	0.0818	0.03434	1.0861	-0.0001	0.0068	-0.0001	0.0042
		$\hat{A}_1$	-0.4161	-0.3554	0.0936	0.0606	1.9182	-0.4161	0.0111	1.7e-05	5.5e-04
		$\hat{B}_1$	0.9093	-0.7741	0.0591	0.1351	4.2740	-0.9092	5.8e-03	6.8e-05	2.1e-03
		$\hat{C}_0$	0.0000	-0.0748	0.0819	-0.0748	2.3664	-0.0003	0.0154	-0.0003	0.0106
		$\hat{C}_1$	-0.9093	0.7710	0.0965	-0.1382	4.3708	0.9089	0.0155	-0.0003	0.0095
		$\hat{D}_1$	-0.4161	-0.3524	0.0453	0.0637	2.0150	-0.4155	0.0091	0.0006	0.0198

Table 6. Simulation results of *LS* and *CLTS* estimators for data with leverage points when a=2

				LS				CLTS			
	n	Estimates	True value	Mean	SE	Bias	RMSE	Mean	SE	Bias	RMSE
10% leverage points	20	$\hat{A}_0$	0.0000	0.1245	0.2478	0.1245	3.9396	0.0009	0.0379	0.0009	0.0289
		$\hat{A}_1$	-0.4161	-0.2381	0.2869	0.1779	5.6279	-0.4152	0.0429	0.0009	0.0285
		$\hat{B}_1$	0.9093	-0.7800	0.1234	0.1292	4.0869	-0.9083	0.0189	0.0009	0.0295
		$\hat{C}_0$	0.0000	-0.4606	0.2144	-0.4606	14.5664	-0.0009	0.0354	-0.0009	0.0311
		$\hat{C}_1$	-0.9093	0.3513	0.2456	-0.5579	17.6423	0.9080	0.0364	-0.0012	0.0403
		$\hat{D}_1$	-0.4161	-0.2964	0.1491	0.1197	3.7860	-0.4156	0.0162	0.0004	0.0152
10% leverage points	50	$\hat{A}_0$	0.0000	0.1294	0.1334	0.1294	4.0940	-7.3e-05	0.0107	-7.3e-05	2.3e-03
		$\hat{A}_1$	-0.4161	-0.2244	0.1548	0.1916	6.0615	-0.4170	0.0153	-0.0008	0.0282
		$\hat{B}_1$	0.9093	-0.7608	0.0760	0.1484	4.6955	-0.9088	0.0090	0.0004	0.0145
		$\hat{C}_0$	0.0000	-0.4313	0.1177	-0.4313	0.1363	-6.9e-05	0.0203	-6.9e-05	2.2e-03
		$\hat{C}_1$	-0.9093	0.3820	0.1390	-0.5272	16.6744	0.9091	0.0208	-0.0001	0.0055
		$\hat{D}_1$	-0.4161	-0.2779	0.0895	0.1382	4.3707	-0.4158	0.0116	0.0003	0.0100
10% leverage points	100	$\hat{A}_0$	0.0000	0.1400	0.0933	0.1400	4.4279	-0.0005	0.0085	-0.0005	0.0173
		$\hat{A}_1$	-0.4161	-0.2132	0.1088	0.2029	6.4167	-0.4168	0.0122	-0.0006	0.0217
		$\hat{B}_1$	0.9093	-0.7584	0.0533	0.1508	4.7690	-0.9092	6.7e-03	4.9e-05	1.5e-03
		$\hat{C}_0$	0.0000	-0.4244	0.0803	-0.4244	13.4238	-0.0011	0.0158	-0.0011	0.0356
		$\hat{C}_1$	-0.9093	0.3869	0.0955	-0.5223	16.5178	0.9083	0.0159	-0.0009	0.0313
		$\hat{D}_1$	-0.4161	-0.2795	0.0574	0.1366	4.3203	-0.4151	0.0094	0.0009	0.0308
20% leverage points	20	$\hat{A}_0$	0.0000	0.1921	0.1981	0.1921	6.0775	0.0011	0.0490	0.0011	0.0376
		$\hat{A}_1$	-0.4161	-0.1209	0.2219	0.2951	9.3338	-0.4153	0.0534	0.0008	0.0255
		$\hat{B}_1$	0.9093	-0.6064	0.1716	0.3028	9.5781	-0.9087	0.0218	0.0005	0.0182
		$\hat{C}_0$	0.0000	-0.5652	0.1688	-0.5652	17.8749	0.0015	0.0306	0.0015	0.0502
		$\hat{C}_1$	-0.9093	0.2152	0.1920	-0.6940	21.9477	0.9106	0.0322	0.0013	0.0436
		$\hat{D}_1$	-0.4161	-0.2074	0.1798	0.2086	6.5990	-0.4163	0.0160	-0.0002	0.0070
20% leverage points	50	$\hat{A}_0$	0.0000	0.2010	0.1152	0.2010	6.3585	0.0001	0.0130	0.0001	0.0049
		$\hat{A}_1$	-0.4161	-0.10492	0.1272	0.3112	9.8417	-0.4166	0.0176	-0.0005	0.0162
		$\hat{B}_1$	0.9093	-0.6027	0.0960	0.3065	9.6948	-0.9083	0.0095	0.0009	0.0300
		$\hat{C}_0$	0.0000	-0.5509	0.0959	-0.5509	17.4240	-0.0015	0.0205	-0.0015	0.0485
		$\hat{C}_1$	-0.9093	0.2309	0.1091	-0.6783	21.4519	0.9072	0.0212	-0.0020	0.0642
		$\hat{D}_1$	-0.4161	-0.1881	0.1030	0.2280	7.2111	-0.4164	0.0112	-0.0003	0.0101
20% leverage points	100	$\hat{A}_0$	0.0000	0.2030	0.0779	0.2030	6.4220	-0.0002	0.0065	-0.0002	0.0093
		$\hat{A}_1$	-0.4161	-0.1007	0.0860	0.3154	9.9744	-0.4165	0.0104	-0.0004	0.0141
		$\hat{B}_1$	0.9093	-0.6000	0.0662	0.3092	9.7796	-0.9092	6.0e-03	7.6e-05	2.4e-03
		$\hat{C}_0$	0.0000	-0.5462	0.0656	-0.5462	17.2733	-0.0006	0.0150	-0.0006	0.0209
		$\hat{C}_1$	-0.9093	0.2331	0.0739	-0.6761	21.3808	0.9086	0.0154	-0.0006	0.0218
		$\hat{D}_1$	-0.4161	-0.1854	0.0727	0.2306	7.2940	-0.4156	0.0094	0.0005	0.0168
30% leverage points	20	$\hat{A}_0$	0.0000	0.2353	0.1522	0.2353	7.4436	0.0043	0.0504	0.0043	0.1387
		$\hat{A}_1$	-0.4161	-0.0248	0.1634	0.3912	12.3731	-0.4119	0.0548	0.0042	0.1333
		$\hat{B}_1$	0.9093	-0.4538	0.1838	0.4554	14.4028	-0.9094	0.0225	-0.0001	0.0047
		$\hat{C}_0$	0.0000	-0.6112	0.1329	-0.6112	0.1932	6.1e-06	0.04637	6.1e-06	1.9e-04
		$\hat{C}_1$	-0.9093	0.15233	0.1455	-0.7569	23.9373	0.9089	0.04771	-0.0003	0.0095
		$\hat{D}_1$	-0.4161	-0.1119	0.1789	0.3042	9.6209	-0.4162	0.0155	-0.0001	0.0032
30% leverage points	50	$\hat{A}_0$	0.0000	0.2326	0.0944	0.2326	7.3559	-0.0003	0.0115	-0.0003	0.0104
		$\hat{A}_1$	-0.4161	-0.0301	0.0997	0.3860	12.2068	-0.4167	0.0151	-0.0005	0.0184
		$\hat{B}_1$	0.9093	-0.4386	0.1126	0.4706	14.8843	-0.9088	0.0091	0.0004	0.0143
		$\hat{C}_0$	0.0000	-0.5985	0.0786	-0.5985	18.9269	-0.0000	0.0181	-0.0000	0.0003
		$\hat{C}_1$	-0.9093	0.1630	0.0870	-0.7461	0.2359	0.9093	0.0190	7.2e-05	2.2e-03
		$\hat{D}_1$	-0.4161	-0.1011	0.1086	0.3149	9.9596	-0.4157	0.0111	0.0004	0.0139
30% leverage points	100	$\hat{A}_0$	0.0000	0.2365	0.0692	0.2365	7.4791	9.2e-05	7.1e-03	9.2e-05	2.9e-03
		$\hat{A}_1$	-0.4161	-0.0260	0.0710	0.3900	0.1233	-0.4160	0.0109	6.7e-05	2.1e-03
		$\hat{B}_1$	0.9093	-0.4398	0.0783	0.4694	14.8445	-0.9089	0.0059	0.0003	0.0098
		$\hat{C}_0$	0.0000	-0.5915	0.0591	-0.5915	18.7055	-0.0005	0.0136	-0.0005	0.0158
		$\hat{C}_1$	-0.9093	0.16660	0.0641	-0.7426	23.4858	0.9088	0.0142	-0.0004	0.0133
		$\hat{D}_1$	-0.4161	-0.0984	0.0716	0.3177	10.0477	-0.4155	0.0084	0.0005	0.0186

Table 7. Results of fitting the *JS* circular regression model for eye data

Parameters	LS estimates	Standard error	CLTS estimates	Standard error
$\hat{A}_0$	1.0821	0.2664	1.2020	0.2000
$\hat{A}_1$	-0.1497	0.1026	-0.1749	0.1013
$\hat{B}_1$	-0.3836	0.2873	-0.4384	0.2172
$\hat{C}_0$	0.0986	0.2776	-1.4845	0.2395
$\hat{C}_1$	0.2533	0.1070	-0.0556	0.0940
$\hat{D}_1$	0.5935	0.2994	1.1307	0.2600
$\hat{\sigma}_1$	0.16	0.1198	0.12	0.1167
$\hat{\sigma}_2$	0.16	0.1635	0.14	0.1396
$A(\hat{\kappa})$	0.9775		0.9951	
$\hat{\kappa}$	22.5056		37.3676	
$\hat{\rho}$	0.9774		0.9827	

increases. Generally, for contaminated data, *LS* leads to a poor estimation of the coefficients. However, the *LS* method is affected by the presence of outliers in the data. The effect worsens with the presence of a higher percentage of contaminated observations in the data. However, the proposed *CLTS* outperform the *LS*.

6. Practical example (eye data)

This section illustrates the proposed *CLTS* on eye data collected from University Malaya Medical Centre. The eye data consist of 23 observations of glaucoma patients (unit in radians) recorded using Optical coherence tomography (OCT) at the University Malaya Medical Centre (UMMC). OCT technology originally is used in ophthalmology to image the posterior segment and has also been used to image anterior segment structures such as the cornea. The angle imaging of the anterior segment OCT in UMMC patients' eyes was obtained with Anterior Segment OCT (AS-OCT). The measurements selected are the angle of the posterior corneal curvature (u) and the angle of the eye (between posterior corneal curvature to iris) (*v*). The Mean Circular Error (*DMCE*) statistic was applied to the data after fitting the *JS* model (Ibrahim, 2013). They showed that there are two vertical outliers with observation numbers 2 and 15. Both *LS* and *CLTS* estimates of parameters are shown in Table 7. We find that the CLTS significantly change the value of ^ A_0, ^ A_1, ^B_1, ^ C_0, ^ C_1, ^D_1, ^$_1$, and ^$_2$ Otherwise, the values of the standard errors for all the parameter estimates of *CLTS* smaller than *LS*. Meanwhile, the estimated concentration parameter has increased from 0.9774 to 0.9827, and A(^) is increased from 0.9775 to 0.9951, as well as ^ increased from 22.5056 to 37.3676. Therefore, the estimation is more accurate, and we may have a better model fitting using the *CLTS* estimator.

7. Conclusion

The least trimmed squares estimator (*LTS*) is a robust regression methods frequently used in practice. Nevertheless, it is not used for JS circular model. This paper introduced the robust estimator *CLTS*, which is robust against vertical outliers and leverage points. The simula tion result illustrated the excellent performance of *CLTS* for contaminated circular data sets. This paper focused on the *LTS* estimation; one might be interested in extending other robust estimation to advanced robust breakdown points estimation methods, such as *M,MM,* or *BS* estimators. Also, this paper has considered a *JS* circular regression model with one independent circular variable; however, future studies might extend the robust *CLTS* estimator in multiple circular regression models.

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